Math 23, Spring 2017

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May 8, 2017

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explicitly, where A is a constant matrix.
- Remark: The solutions of an IVP associated to $X' = A \cdot X$ are unique and defined in $\mathbb{R}$

Example 0

Solve

$$X' = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \cdot X$$

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What are the eigenvalues and the eigenvectors of $\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$?

Exponential solutions

We know how to solve

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Let's search for solutions of the type $X(t) = e^{\lambda t} v$.

Eigenvalues and Eigenvectors

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Very similar to solving $ay'' + by' + cy = 0$.

Case (A), all eigenvalues are real and distinct

- A $n \times n$ matrix
- $\lambda_1, \ldots, \lambda_n$ distinct eigenvalues
- $v_1, \ldots, v_n$ corresponding eigenvectors

We want to show that $\{e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n\}$ is a fundamental solution set.

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What is the value of the Wronksian at $t = 0$? Indeed,

$$W(e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n)(t) = e^{(\lambda_1 + \cdots + \lambda_n)t} \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

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Claim

$$v_1, \ldots, v_n \text{ are linearly independent} \iff \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \neq 0$$

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$v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v_{-1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.