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Edgar Costa

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Dartmouth College

Definition

A system of ODEs is **linear**, if it can be written as:

$$\begin{cases} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_1 + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_2 + g_2(t) \\ \vdots &= \quad \vdots \qquad \ddots \qquad \vdots \qquad + \quad \vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases} \iff X' = P(t) \cdot X + G(t)$$

Where

$$X = (x_1, \ldots, x_n)^T \qquad P(t) = \{p_{ij}(t)\}_{i,j} \qquad G(t) = (g_1(t), \ldots, g_n(t))^T$$

If $G(t) = 0$ (i.e, $g_i = 0$), then this system is called a **homogeneous** system.

Theorem 7.4.1

If ϕ_1 and ϕ_2 are two solutions for

$$X' = P(t)X,$$

then $c_1\phi_1 + c_2\phi_2$ is also a solution.

Proof?

Independent solutions

Definition

Let $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ be n solutions to an n-dimensional homogeneous linear system

$$X' = P(t) \cdot X.$$

The solutions $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ are linearly independent at a point t_0 if

$$W(\phi_1, \phi_2, \ldots, \phi_n)(t_0) := \det \begin{pmatrix} | & & | \\ \phi_1(t_0) & \cdots & \phi_n(t_0) \\ | & & | \end{pmatrix} \neq 0$$

$W(\phi_1, \phi_2, \ldots, \phi_n)(t)$ is called the **Wronskian** of the n-solutions.

General homogeneous solution (compare with Theorem 3.2.4)

Theorem 7.4.2

Consider the n-dimensional homogeneous linear system

$$X' = P(t) \cdot X$$

with $P : (\alpha, \beta) \to \mathbb{R}^{n^2}$ continuous.

If $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$ are solutions and linear independent at some point t_0, then every solution in (α, β) is of the shape

$$c_1\phi_1(t) + \cdots + c_n\phi_n(t).$$

Proof: Given another solution, Ψ, construct an IVP at t_0 such that $c_1\phi_1(t) + \cdots + c_n\phi_n(t)$ also solves it for some $c_1, \dots, c_n$.

A corollary

If $W(\phi_1, \phi_2, \ldots, \phi_n)(t_0) \neq 0$, then **any** IVP in (α, β) as a solution of the shape

$$c_1\phi_1(t) + \cdots + c_n\phi_n(t).$$

Which means that for any $t \in (\alpha, \beta)$ the following system has a unique solution

$$\begin{pmatrix} | & & | \\ \phi_1(t) & \cdots & \phi_n(t) \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = X_0$$

Hence, $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Theorem 7.4.3 (Compare with Abel's Theorem)

Either $W(\phi_1, \phi_2, \ldots, \phi_n)(t)$ is never 0, or is always identically 0 in (α, β).

Consider the n-dimensional homogeneous linear system

$$X' = P(t) \cdot X$$

with $P : (\alpha, \beta) \to \mathbb{R}^{n^2}$ continuous.

Definition

If $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$ for some $t \in (\alpha, \beta)$ ($\Rightarrow$ for all t), then $\{\phi_1, \ldots, \phi_n\}$ is called a **fundamental solution set**.

Particular solutions

Theorem

If ϕ_1 and ϕ_2 are solutions for

$$X' = P(t) \cdot X + G(t),$$

then $\phi_1 - \phi_2$ are solutions for

$$X' = P(t) \cdot X.$$

Thus, to generically solve $X' = P(t) \cdot X + G(t)$ we need to

- find n linear independent solutions for the homogeneous system $X' = P(t) \cdot X$.
- find a particular solution for the system $X' = P(t) \cdot X + G(t)$.

Exercise 7.4.7

Consider $\phi_1 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ $\phi_2 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

1. Compute $W(\phi_1, \phi_2)(t)$
2. In what intervals are ϕ_1 and ϕ_2 linear independent.
3. What conclusions can be made about the system of linear homogeneous ODE that ϕ_1 and ϕ_2 satisfy.
4. Figure out the system. (warning: algebra heavy)