Math 23, Spring 2017

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$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_n \\ \vdots & & \vdots & & \ddots & & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right. \iff A \cdot x = b,$$

$$\text{where } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Definition

If $b = 0$ then the systems of linear equations is called **homogeneous**.

Determinant

The system of linear equations

$$Ax = b$$

as **exactly** one solution if and only if $\det A \neq 0$ (we will discuss det later on) and A is called **nonsingular**.

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If $\det A = 0$, then A is called **singular**, and either

- there are no solutions (think two parallel lines that do not intersect)
- or, there are infinitely many solutions (think two parallel lines that coincide)

Augmented matrix

Definition

The **augmented matrix** of the system $Ax = b$ is

$$A \mid b = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix} \text{ it is a } m \times (n+1) \text{ matrix.}$$

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We can solve the system with elementary operations on the augmented matrix.

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   $\Leftrightarrow$ exchanging the order of the equations
2. adding to a row a multiple of **another** row

   $\Leftrightarrow$ multiplying one equation and adding it to another equation

Elementary Operations

1. exchanging **rows**
   $\Leftrightarrow$ exchanging the order of the equations
2. adding to a row a multiple of **another** row
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Applying the operations above to $A \,|\, b$ does not change the solutions of $Ax = b$.

Exercise 7.3.2

Solve

$$\begin{cases} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 1 \end{cases}$$

Elementary Operations

The general goal we are trying to reach is to reduce the augmented matrix to a matrix

- all nonzero rows are above any rows of all zeroes, and
- the leading coefficient of a nonzero row is always strictly to the right of the leading coefficient of the row above it

Example

$$\begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \end{pmatrix}$$

where $\blacksquare$ means $\neq 0$.

Examples in more detail

- $\begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ has **infinitely** many solutions.

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- $\begin{pmatrix} 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \end{pmatrix}$ has **no** solutions.

Examples in more detail

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- $\begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{pmatrix}$ has **exactly** one solution.

Determinants

Only for square matrices

- $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

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- $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
- For 3×3 there are many "ways" to compute:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = (aei + bfg + cdh) - (afh + bdi + ceg)$$

$$= a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Determinants

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$$= -b\det\begin{pmatrix} \otimes & \otimes & \otimes \\ d & \otimes & f \\ g & \otimes & i \end{pmatrix} + e\det\begin{pmatrix} a & \otimes & c \\ \otimes & \otimes & \otimes \\ g & \otimes & i \end{pmatrix} - h\det\begin{pmatrix} a & \otimes & c \\ d & \otimes & f \\ \otimes & \otimes & \otimes \end{pmatrix}$$

Determinants

$$\det\begin{pmatrix} a^{\oplus} & b^{\ominus} & c^{\oplus} \\ d^{\ominus} & e^{\oplus} & f^{\ominus} \\ g^{\oplus} & h^{\ominus} & i^{\oplus} \end{pmatrix} = (aei + bfg + cdh) - (afh + bdi + ceg)$$

$$= a\det\begin{pmatrix} \otimes & \otimes & \otimes \\ \otimes & e & f \\ \otimes & h & i \end{pmatrix} - b\det\begin{pmatrix} \otimes & \otimes & \otimes \\ d & \otimes & f \\ g & \otimes & i \end{pmatrix} + c\begin{pmatrix} \otimes & \otimes & \otimes \\ d & e & \otimes \\ g & h & \otimes \end{pmatrix}$$

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Determinant 4x4 example

Using the last row (in both matrices)

$$\det \begin{pmatrix} 1^{\oplus} & 2 & 3 & 4 \\ 0^{\ominus} & 1 & 2 & 3 \\ 0^{\oplus} & 0 & 1 & 2 \\ 0^{\ominus} & 0^{\oplus} & 0^{\ominus} & 1^{\oplus} \end{pmatrix} = -0\det(3\times 3) + 0\det(3x3) - 0\det(3x3) + 1\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\text{and}\ \ \det\begin{pmatrix} 1^{\oplus} & 2 & 3 \\ 0^{\ominus} & 1 & 2 \\ 0^{\oplus} & 0^{\ominus} & 1^{\oplus} \end{pmatrix} = 0\det(2\times 2) - 0\det(2\times 2) + 1\det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= 0 - 0 + 1(1\cdot 1 - 2\cdot 0) = 1$$

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$$= 0 - 0 + 1(1\cdot 1 - 2\cdot 0) = 1$$

Therefore, $\det(4\times 4) = 1$

Checkout the KA video: link

Exercise 7.3.24

Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

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Definition

- The **characteristic polynomial of** A is $\det(A - \lambda I)$
- **eigenvalues** are the roots of the characteristic polynomial of A
- **eigenvectors** corresponding to an eigenvalue $\tilde{\lambda}$ are the vector solutions of

$$(A - \tilde{\lambda} I)v = 0 \Leftrightarrow Av = \tilde{\lambda} v$$

Solution Exercise 7.3.24 (eigenvalues)

$$
\begin{aligned}
\det(A-\lambda I) &= \det\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \\
&= (3-\lambda)\det\begin{pmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} - 2\det\begin{pmatrix} 2 & 4 \\ 2 & 3-\lambda \end{pmatrix} + 4\det\begin{pmatrix} 2 & 4 \\ -\lambda & 2 \end{pmatrix} \\
&= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda+1)^2(\lambda-8)
\end{aligned}
$$

The eigenvalues are -1 and 8.

Solution Exercise 7.3.24 (eigenvectors for $\lambda = -1$)

$$(A - (-1)I)v = 0 \Leftrightarrow \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

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Row reduce the augmented matrix $\begin{pmatrix} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

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$\Leftrightarrow 2x_1 + x_2 + 2x_3 = 0 \Leftrightarrow x_1 = -\frac{1}{2}x_2 - x_3.$

In terms of vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}}_{v_1} x_2 + \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{v_2} x_3$

Solution Exercise 7.3.24 (eigenvectors for $\lambda = -1$)

$v_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are both eigenvectors for $\lambda = -1$

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$v_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are both eigenvectors for $\lambda = -1$

Any linear combination of them is also an eigenvector, just check it:

$$A \cdot (c_1 v_1 + c_2 v_2) = c_1 A \cdot v_1 + c_2 A \cdot v_2 = c_1 \lambda v_1 + c_2 \lambda v_2 = \lambda (c_1 v_1 + c_2 v_2)$$

Solution Exercise 7.3.24 (eigenvectors for $\lambda = 8$)

$$(A - 8I)v = 0 \Leftrightarrow \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

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Row reduce the augmented matrix $\begin{pmatrix} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -4 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\begin{cases} x_1 - 4x_2 + x_3 = 0 \\ 2x_2 + x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 2x_2 \\ x_3 = 2x_2 \end{cases} \Leftrightarrow v = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} x_2$$

Linear independence

We say that $b_1 = \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix}, \cdots, b_k = \begin{pmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{pmatrix}$ are **linearly dependent** if exists $c_1, \cdots, c_k$ not all zeros such that

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If $k = n$, then linearly independence can be checked by computing the determinant of the matrix above!