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So far we have been studying one equation at a time.

What happens if there is more than one force in play? How can we handle that.

n-body problem

Given position, velocity and time of a group of celestial bodies, predict their interactive forces; and consequently, predict their true orbital motions for all future times.

We can replace "Gravity" by other forces...

Check out the applet:
`http://www.princeton.edu/~rvdb/WebGL/nBody.html`

$$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right)$$

can be rewritten as

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} := \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix} \quad \rightsquigarrow \quad X'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \\ F(t, x_1, x_2, \ldots, x_n) \end{pmatrix}$$

The generic system of ODE's

In general we are interested in studying

$$\begin{cases} x_1' &= F_1(t, x_1, \ldots, x_n) \\ \vdots &= \quad \vdots \\ x_n' &= F_n(t, x_1, \ldots, x_n) \end{cases} \qquad (\Delta)$$

A solution for (Δ) is a collection of n differentiable functions $\phi_i(t)$ in $t \in (\alpha, \beta)$, with $i = 1, \ldots, n$, such that taking $x_i = \phi_i$ makes all the equations in (Δ) true identities.

We have an Initial Value Problem if we are also given initial conditions of the form

$$x_1(t_0) = a_0, \; \cdots \; , x_n(t_0) = a_n$$

Existence and uniqueness

Theorem 7.1.1 (generalization of Theorem 2.4.1)

Consider the following Initial Value Problem:

$$\begin{cases} x_1' &= F_1(t, x_1, \dots, x_n), \quad x_1(t_0) = a_0 \\ \vdots &= \quad \vdots \\ x_n' &= F_n(t, x_1, \dots, x_n), \quad x_n(t_0) = a_n \end{cases} \qquad (\Delta)$$

Assume that F_i and $\frac{\partial F_i}{\partial x_j}$ (but not necessarily $\frac{\partial F_i}{\partial t}$) are **continuous** in a small $(n+1)$-dimensional box containing $(t_0, a_0, \dots, a_n)$.

Then there exists $h > 0$ (probably very small) such that the (Δ) has a **unique** solution on $(t_0 - h, t_0 + h)$.

Q: What about linear systems? What is a linear system?

First order linear systems of ODEs

Definition

The system (Δ) is **linear**, if F_i are linear functions of x_j, i.e., if it can be written as:

$$\begin{cases} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_1 + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_2 + g_2(t) \\ \vdots &= \quad \vdots \qquad\quad \ddots \qquad \vdots \qquad + \quad \vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases} \Leftrightarrow X' = P(t) \cdot X + G(t)$$

Where

$$X = (x_1, \ldots, x_n)^T \qquad P(t) = \{p_{ij}(t)\}_{i,j} \qquad G(t) = (g_1(t), \ldots, g_n(t))^T$$

If $G(t) = 0$ (i.e, $g_i = 0$), then this system is called a **homogeneous** system.

Existence and uniqueness for linear systems

Theorem 7.1.2

If $P(t) : (\alpha, \beta) \to \mathbb{R}^{n^2}$ and $G(t) : (\alpha, \beta) \to \mathbb{R}^n$ are continuous, then there is a unique solution, defined in (α, β) for the Initial Value Problem (with $t_0 \in (\alpha, \beta)$)

$$X' = P(t) \cdot X + G(t), \quad X(t_0) = X_0$$

An $m \times n$ matrix A is a "table" with m rows and n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We may also write $A = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$

We may also refer to the element in the ith row and jth column by A_{ij}.

Transpose

Definition

The **transpose** of A, A^T or A^t, is a $n \times m$ matrix, obtained by interchanging the rows and columns, i.e., $\left(A^T\right)_{ij} = A_{ji}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 1+i & 2 & 3 \\ -i & \pi & e-i \end{pmatrix}$. Compute A^T.

Conjugate

Definition

The **conjugate** of A, $\overline{A}$, is a $m \times n$ matrix, obtained by conjugating all entries of A, i.e, $\left(\overline{A}\right)_{ij} = \overline{A_{ij}}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \overline{A} = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \cdots & \overline{a_{1n}} \\ \overline{a_{21}} & \overline{a_{22}} & \cdots & \overline{a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{m1}} & \overline{a_{m2}} & \cdots & \overline{a_{mn}} \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 1+i & 2 & 3 \\ -i & \pi & e-i \end{pmatrix}$. Compute $\overline{A}$.

Adjoint

Definition

We defined the **adjoint** of A as $A^* := \overline{A^T} = \overline{A}^T$.

Exercise

Given $A = \begin{pmatrix} 1+i & 2 & 3 \\ -i & \pi & e-i \end{pmatrix}$. Compute A^*.

Operations on matrices: Equality, addition and subtraction

Equality: two matrices are equal if they have the same dimensions and the same entries.

Addition and subtraction: it's done entry wise.

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

Note: $A+B$ and $A-B$ are only defined if they have the same dimensions.

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}$

Compute $A+B$, $A-B$ and $A+B^T$.

Operations on matrices: Zero matrix

Definition

The **zero matrix** is a $m \times n$ matrix with all entries zero.

Example

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Operations on matrices: Scalar multiplication

Scalar multiplication: it's done entry wise

$$(\alpha A)_{ij} = \alpha A_{ij}, \quad \alpha \in \mathbb{R} \text{ or } \mathbb{C}$$

Example

$$3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

So far, **addition**, **subtraction**, **scalar multiplication** and the **zero matrix**, behave very similarly to vectors, except we represent them as tables instead of lists.

Matrix multiplication

Remark

Matrices can be multiplied **only** if the first matrix has as many columns as the second matrix has rows.

- A is a $m \times n$ matrix
- B is a $n \times k$ matrix
- then $A \cdot B$ is a $m \times k$ matrix

The element in the ith row and jth column of $A \cdot B$ is

$$(A \cdot B)_{ij} = \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$

In other words, we have to transverse the ith row of A and the jth column of B and sum the product of the corresponding entires.

Matrix multiplication

$(A \cdot B)_{ij} = \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$

Example

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 + 1 \cdot 2 & 1 \cdot 0 + 2 \cdot 1 + 1 \cdot 3 \\ 3 \cdot 1 + 2 \cdot 0 + 3 \cdot 2 & 3 \cdot 0 + 2 \cdot 1 + 3 \cdot 3 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 9 & 11 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 11 & 10 & 11 \end{pmatrix} \neq A \cdot B$$

Matrix multiplication is not commutative! (not even of the same shape)

Multiplication properties

Sometimes $A \cdot B$ exists, but $B \cdot A$ does not. Example?

It is **associative** and **distributive** (if all the operations are defined)

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$(B + C) \cdot A = B \cdot A + C \cdot A$$

Identity matrix

The **Identity** matrix is a $n \times n$ matrix, with 1s on the diagonal and zeros everywhere else.

$$I_{n\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

If A is a $m \times n$ matrix

$$A \cdot I_{n\times n} = A = I_{m\times m} \cdot A$$

Vectors as matrices

An n-dimensional vector can be interpreted as a row, i.e., $1 \times n$ matrix, or as a column, i.e., $n \times 1$ matrix.

Example: inner product

$$x, y \in \mathbb{R}^n,\ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Then $< x, y >= \sum_{i=1}^{n} x_i y_i = x^T \cdot y$

Exercise

Use matrix multiplication to compute the norm of $x = (1\,2\,1\,3)^T$.

Hint: $||x|| = \sqrt{< x, x >}$.