Math 23, Spring 2017

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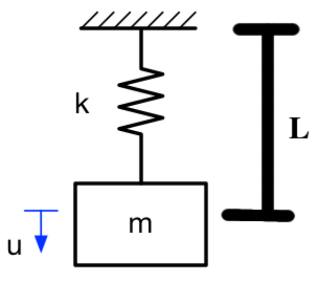
Spring mass system

- Gravity force $= mg$
- Spring force $= -k\ell$, where ℓ is the elongation of the spring
- $L =$ the equilibrium position, i.e., $mg = KL$
- $u(t) =$ the displacement of mass from the equilibrium position, measured downwards
- $\Rightarrow$ Spring force $= -k(L + u)$
- Damping/Friction: $F_d = -\gamma u'$ (opposite direction)
- External force: $F(t)$



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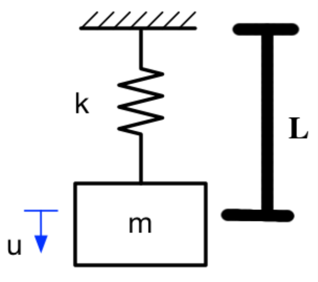

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We can also do the same to model electric circuits.

Classification

$$mu'' + \gamma u' + ku = F(t)$$

$$m, k > 0 \text{ and } \gamma \geq 0$$

$$mr^2 + \gamma r + k = 0 \Rightarrow r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

- $\gamma = 0$ undamped
- $\gamma > 0$ damped
  - $\gamma^2 - 4km < 0$ underdamped
  - $\gamma^2 - 4km = 0$ critically damped (no oscillation)
  - $\gamma^2 - 4km > 0$ overdamped (no oscillation)
- $F(t) = 0$ free
- $F(t) \neq 0$ and periodic, forced vibration

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- $\gamma^2 - 4km < 0$ undamped or underdamped $\Rightarrow r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = -\alpha \pm \omega_0 i$
  $u(t) = e^{-\alpha t}(c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)) = \cdots = Ae^{-\alpha t}\cos(\omega_0 t - b)$
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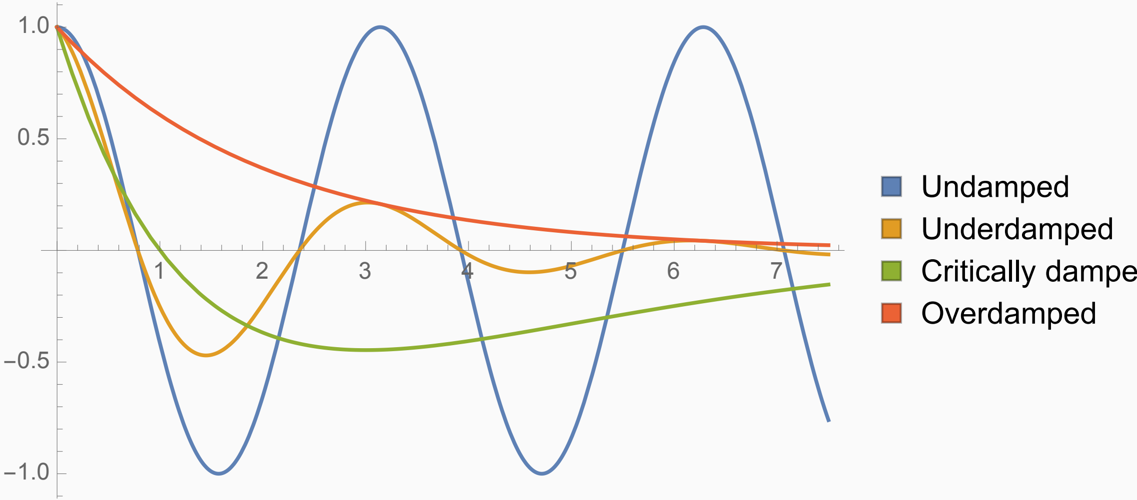
- $\gamma^2 - 4km > 0$ overdamped, $u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, with $r_1, r_2 < 0$

A picture is worth a thousand formulas



Forced vibrations with damping

$$mu'' + \gamma u' + ku = F_0 \cos(\omega_0 t)$$
$$m, k > 0 \text{ and } \gamma > 0$$

$u(t) = u_H(t) + u_p(t)$

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If $\gamma > 0$,

$$\lim_{t \to +\infty} u_H(t) = 0$$
$$u_p(t) = A\cos(\omega_0 t) + B \sin(\omega_0 t)$$

Thus, regardless of the IVP if $\gamma > 0$ $u(t) \simeq u_p(t)$ as $t \to +\infty$.

Resonance

$\omega_0 = k/m$ and $\gamma = 0$ (for simplicity)

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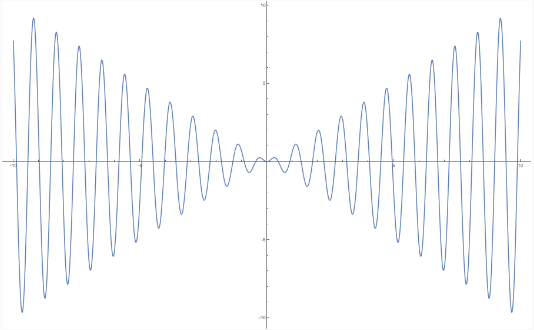

Resonance with damping

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

$$m, k > 0 \text{ and } \gamma > 0$$

$$u_p(t) = R\cos(wt - b_0)$$

Picking $\omega^2 = \frac{k}{m}\left(1 - \frac{\gamma^2}{2mk}\right)$ and γ small, we get $R \simeq \frac{F_0 m}{\gamma\sqrt{k/m}}\left(1 + \frac{\gamma^2}{8mk}\right)$.

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Thus, if $\gamma\sqrt{k/m} \to 0$, the amplitude of particular solution goes to $+\infty$.

Exercise 3.7.7

A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.