Math 23, Spring 2017

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Last time: Method of undetermined coefficients

To find the general solution for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{NH}$$

we need to find **one** solution for it, y_p known as a **particular solution**, and then find a fundamental solution set $\{y_1, y_2\}$ for the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{H}$$

Then, we can write the general solution for (NH) as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

Note: Only works for a narrow class of functions $g(t)$

Today: Variation of parameters

Idea

Assume that $\{y_1, y_2\}$ is a fundamental solution set for

$$y'' + p(t)y' + q(t)y = 0. \tag{H}$$

What if we search for $y_p(t)$, a solution of

$$y'' + p(t)y' + q(t)y = g(t), \tag{NH}$$

of the following form

$$\tilde{y}(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

with $u_1(t)$ and $u_2(t)$ unknown.

Today: Variation of parameters

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with $u_1(t)$ and $u_2(t)$ unknown.

Note: Very similar to the reduction of order method!

Variation of parameters: some assumptions

We have

$$\tilde{y} = u_1 y_1 + u_2 y_2$$
$$\tilde{y}' = u_1' y_1 + u_2' y_1 + u_1 y_1' + u_2 y_2'$$

If we don't impose anything extra condition on u_1 and u_2, then $\tilde{y}''$ will depend on u_1'' and u_2'', and this will lead to a harder problem!

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If we don't impose anything extra condition on u_1 and u_2, then $\tilde{y}''$ will depend on u_1'' and u_2'', and this will lead to a harder problem! We impose

$$u_1' y_1 + u_2' y_2 = 0 \text{ for all } t$$

Thus

$$\tilde{y}'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

Variation of parameters: the math

Plugging

$$\begin{aligned} \tilde{y} &= u_1 y_1 + u_2 y_2 \\ \tilde{y}' &= u_1' y_1 + u_2' y_1 + u_1 y_1' + u_2 y_2' \\ \tilde{y}'' &= u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''. \end{aligned}$$

in $y'' + p(t)y' + q(t)y = g(t)$, we get

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in $y'' + p(t)y' + q(t)y = g(t)$, we get two equations

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1(y_1'' + py_1' + qy_1') + u_2(y_2'' + py_2' + qy_2') + u_1' y_1' + u_2' y_2' = g(t) \end{cases} \Leftrightarrow \begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases}$$

Can we solve the system?

The return of the Wronskian

We started with a fundamental solution set, thus $W(y_1, y_2)(t) \neq 0$. Hence, we can solve for u_1' and u_2' algebraically.

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases} \quad \Leftrightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

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Recall, if $ad - bc \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} \Leftrightarrow \begin{cases} x = \frac{1}{ad-bc}(de - bf) \\ y = \frac{1}{ad-bc}(af - ce) \end{cases}$$

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In our case,

$$\begin{cases} u_1' = \frac{-gy_2}{W(y_1,y_2)} \\ u_2' = \frac{gy_1}{W(y_1,y_2)} \end{cases} \Leftrightarrow \begin{cases} u_1 = \int \frac{-gy_2}{W(y_1,y_2)} \\ u_2 = \int \frac{gy_1}{W(y_1,y_2)} \end{cases}$$

Upshot

Variation of parameters

Assume that $\{y_1, y_2\}$ is a fundamental solution set for $y'' + p(t)y' + q(t)y = 0$.

Then,

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t), \quad \text{with} \begin{cases} u_1 = \int \frac{-gy_2}{W(y_1,y_2)} \\ u_2 = \int \frac{gy_1}{W(y_1,y_2)} \end{cases}$$

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Exercise 3.6.14

Consider the DE $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$

1. Verify that $\{t, te^t\}$ is a fundamental solution set.
2. Find a particular solution.

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Consider the DE $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$

1. Verify that $\{t, te^t\}$ is a fundamental solution set.
2. Find a particular solution.

Note: It would have been enough to know that t solves the homogeneous eqn.

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Exercise 3.6.5

Solve $y'' + y = \tan(t)$