Math 23, Spring 2017

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Nonhomogeneous equations

Last week we build an approach how to solve

$$y'' + p(t)y' + q(t)y = 0.$$

What about

$$y'' + p(t)y' + q(t)y = g(t)?$$

Theorem 3.5.1

If $\tilde{y}_1$ and $\tilde{y}_2$ are solutions of

$$y'' + p(t)y' + q(t)y = g(t), \tag{NH}$$

then $\tilde{y}_1 - \tilde{y}_2$ is a solution of

$$y'' + p(t)y' + q(t)y = 0 \tag{H}$$

Corollary

Assume that $\{y_1, y_2\}$ is a fundamental solution set of (H). and y_p is a **particular solution** of (NH). Then every solution for (NH) can be written as

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t).$$

Method of undetermined coefficients

To find the general solution for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{NH}$$

we need to find **one** solution for it, y_p known as a **particular solution**, and then find a fundamental solution set $\{y_1, y_2\}$ for the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{H}$$

Then, we can write the general solution for (NH) as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

Some particular solutions for $c_n y^{(n)} + \cdots + c_1 y' + c_0 y = g(t) \quad (\Delta)$

Write $p(r) := c_n r^n + \cdots + c_1 r + c_0$

$g(t)$ — RHS of (Δ)	$y_p(t)$ — a particular solution for (Δ)
$a_k t^k + \cdots + a_1 t + a_0$	$t^s(A_k t^k + \cdots + A_1 t + A_0)$ s = number of times 0 is a root of $p(r)$
$e^{\alpha t}(a_k t^k + \cdots + a_1 t + a_0)$	$t^s e^{\alpha t}(A_k t^k + \cdots + A_1 t + A_0)$ s = number of times α is a root of $p(r)$
$e^{\alpha t}\cos(\beta t)(a_k t^k + \cdots + a_1 t + a_0)$ $+$ $e^{\alpha t}\sin(\beta t)(b_k t^k + \cdots + b_1 t + b_0)$ Note: we can have a_i or $b_i = 0$	$e^{\alpha t} t^s \cos(\beta t)(A_k t^k + \cdots + A_1 t + A_0)$ $+$ $e^{\alpha t} t^s \sin(\beta t)(B_k t^k + \cdots + B_1 t + B_0)$ s = number of times $\alpha + i\beta$ is a root of $p(r)$

Example: $y'' + y' + y = g(t)$

$$y'' + y' + y = g(t)$$

- Homogeneous equation: $y'' + y' + y = 0$
- Characteristic equation: $r^2 + r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
- General solution for the homogeneous equation:

$$y(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)$$

- A general solution for $y'' + y' + y = g(t)$ will be of the shape:

$$y(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + y_p(t)$$

where $y_p(t)$ **depends** on $g(t)$.

Example: $y'' + y' + y = t^3 + 1$

- $g(t) = t^3 + 1 \rightsquigarrow y_p(t) = t^s(A_3t^3 + A_2t^2 + A_1t + A_0)$ for some A_3, A_2, A_1, A_0
- $s = 0$, as 0 is **not** a root of $r^2 + r + 1$ ($0^2 + 0^1 + 1 \neq 0$)
- Thus $y_p(t)$ is of the shape $A_3t^3 + A_2t^2 + A_1t + A_0$
- $y_p'' + y_p' + y_p = t^3 + 1 \Rightarrow$

$$(6A_3t + 2A_2) + (3A_3t^2 + 2A_2t + A_1) + (A_3t^3 + A_2t^2 + A_1t + A_0) = t^3 + 1$$

$$\Leftrightarrow$$

$$A_3t^3 + (3A_3 + A_2)t^2 + (6A_3 + 2A_2 + A_1)t + (2A_2 + A_1 + A_0) = 1t^3 + 0t^2 + 0t + 1$$

$$\Leftrightarrow \begin{cases} 1 & = A_3 \\ 0 & = 3A_3 + A_2 \\ 0 & = 6A_3 + 2A_2 + A_1 \\ 1 & = 2A_2 + A_1 + A_0 \end{cases} \Leftrightarrow \begin{cases} A_3 & = 1 \\ A_2 & = -3 \\ A_1 & = 0 \\ A_0 & = 7 \end{cases}$$

Example: $y'' + y' + y = t^3 + 1$

Answer: A general solution for

$$y'' + y' + y = t^3 + 1$$

is

$$y = \underbrace{c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)}_{\text{the general solution for the homogeneous equation}} + \underbrace{t^3 - 3t^2 + 7}_{\text{a particular solution}}$$

$y'' + y' + y = e^{-t/2}(t^2 + 3)$

- $g(t) = e^{-t/2}(t^2 + 3) \rightsquigarrow y_p(t) = e^{-t/2} t^s (A_2 t^2 + A_1 t + A_0)$ for some A_2, A_1, A_0
- $s = 0$, as $-\frac{1}{2}$ is **not** a root of $r^2 + r + 1$
- Thus $y_p(t)$ is of the shape $e^{-t/2}(A_2 t^2 + A_1 t + A_0)$
- Plugging it all in and solving the system we get

$$A_0 = \frac{4}{9},\ A_1 = 0,\ A_2 = \frac{12}{9}$$

- Thus the general solution for $y'' + y' + y = e^{-t/2}(t^2 + 3)$ is

$$y(t) = \underbrace{c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)}_{\text{the general solution for the homogeneous equation}} + \underbrace{e^{-t/2} \frac{12t^2 + 4}{9}}_{\text{a particular solution}}$$

Example: $y'' + y' + y = \cos(\sqrt{3}t)t^2$

$$y'' + y' + y = \cos(\sqrt{3}t)t^2$$

- $0 + \sqrt{3}i$ is not a root of $r^2 + r + 1 \Rightarrow s = 0$

$$y_p(t) = e^{0t}\cos(\sqrt{3}t)(A_2t^2 + A_1t + A_0) + e^{0t}\sin(\sqrt{3}t)(B_2t^2 + B_1t + B_0)$$

- a system in six unknowns and six equations...

$$A_2 = \frac{49}{343}\sqrt{3} \qquad B_2 = \frac{-98}{343}$$

$$A_1 = \frac{84}{343}\sqrt{3} \qquad B_1 = \frac{322}{343}$$

$$A_0 = \frac{-222}{343}\sqrt{3} \qquad B_0 = \frac{-18}{343}$$

Example: $y'' + y' + y = e^{-t/2}\cos(\sqrt{3}t/2)(t+1)$

$$y'' + y' + y = e^{\frac{-t}{2}} \cos\left(\frac{\sqrt{3}t}{2}\right)(t+1)$$

- $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$ is a **simple** root of $r^2 + r + 1 \rightsquigarrow s = 1$
- $$y_p(t) = te^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)(A_1 t + A_0) + te^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)(B_1 t + B_0)$$

Example: $y'' + 4y' + 3y = g(t)$

- $r^2 + 4r + 3 = (r+1)(r+3)$, thus the general homogeneous solution is
$$y_H(t) = c_1 e^{-3t} + c_2 e^{-t}$$
- $y'' + 4y' + 3y = e^{-3t}(t+3)$
  $r = -3$ is simple root $\rightsquigarrow$ $s = 1$ and
$$y_p(t) = e^{-3t} t (A_1 t + A_0)$$
- $y'' + 4y' + 3y = e^{4t}(t+3)$
  $r = 4$ is not a root $\rightsquigarrow$ $s = 0$ and
$$y_p(t) = e^{4t}(A_1 t + A_0)$$
- $y'' + 4y' + 3y = \cos(2t)$
  $0 + 2i$ is not a root $\rightsquigarrow$ $s = 0$ and
$$y_p(t) = \cos(2t) A_0 + \sin(2t) B_0$$

Example: $y'' - 6y' + 9y = g(t)$

- $r^2 - 6r + 9 = (r-3)^2$, thus the general homogeneous solution is
$$y_H(t) = c_1 e^{3t} + c_2 t e^{3t}$$
- $y'' - 6y' + 9y = e^{3t}(t+2)$
  $r = 3$ is double root $\rightsquigarrow s = 2$ and
$$y_p(t) = e^{3t} t^2 (A_1 t + A_0)$$
- $y'' - 6y' + 9y = e^{3t}\cos(t)(t+2)$
  $r = 3 + i$ is not a root $\rightsquigarrow s = 0$ and
$$y_p(t) = e^{3t}\cos(t)(A_1 t + A_0) + e^{3t}\sin(t)(B_1 t + B_0)$$

Example: $y'' - 6y' + 9y = g(t)$

- $y'' - 6y' + 9y = t + 1 \rightsquigarrow$
$$y_p(t) = \frac{1}{9}t + \frac{5}{27}$$
- $y'' - 6y' + 9y = 3\cos(2t) \rightsquigarrow$
$$y_p(t) = \frac{15}{169}\cos(2t) + \frac{-36}{169}\sin(2t)$$
- $y'' - 6y' + 9y = t + 1 + 3\cos(2t) \rightsquigarrow$
$$y_p(t) = \underbrace{\frac{1}{9}t + \frac{5}{27}}_{\text{particular solution for } g(t)=t+1} + \underbrace{\frac{15}{169}\cos(2t) + \frac{-36}{169}\sin(2t)}_{\text{particular solution for } g(t)=3\cos(2t)}$$

Superposition of particular solutions

Theorem

If y_1 is a solution of

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and y_2 is a solution of

$$y'' + p(t)y' + q(t)y = g_2(t),$$

then $y_1 + y_2$ is a solution of

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t).$$