Series Solutions Near a Regular Singular Point

Suppose we have the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

where $x = 0$ is a regular singular point. This means that the limits

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)} \tag{1}$$

and

$$q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)} \tag{2}$$

exist. Sometimes, it is convenient to rewrite this in the form (by dividing by $P(x)$ and multiplying by x^2):

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0. \tag{3}$$

Our initial guess for a solution to this equation is

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

where $a_0 \neq 0$. After taking derivatives and plugging y' and y'' into equation (3), we can find the **indicial equation** $F(r) = 0$ by looking at the coefficient of x^r, which is equal to $a_0F(r)$. The indicial equation can also be found directly from the equation:

$$F(r) = r(r-1) + p_0r + q_0 = 0 \tag{4}$$

where p_0 and q_0 are the limits in equations (1) and (2). The roots of $F(r)$, denoted by r_1 and r_2, are called the **exponents at the singularity**. Suppose $r_1 \geq r_2$ and are real. Then in the interval $0 < x < \rho$, where ρ is the minimum of the radii of convergence, there is a solution of the form:

$$y_1(x) = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right] \tag{5}$$

where $a_n(r_1)$ is obtained from the recurrence relation one gets from looking at the coefficient of x^{n+r} and plugging in $r = r_1$ and $a_0 = 1$.
There are three possibilities for the second solution:

- Case 1: $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer. In this case, there is a second solution of the form:

  $$y_2(x) = x^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right].$$

  where $a_n(r_2)$ is also found by the same recurrence relation and setting $r = r_2$ and $a_0 = 1$.

- Case 2: $r_1 = r_2$. In this case the second solution is of the form:

$$y_2(x) = y_1(x)\ln(x) + x^{r_2}\left[1 + \sum_{n=1}^{\infty} b_n(r_2)x^n\right].$$

  Here, $b_n(r_2)$ is equal to the derivative of a_n with respect to r evaluated at $r_2 = r_1$.

- Case 3: $r_1 - r_2 = N$ is a positive integer. In this case the second solution is of the form:

$$y_2(x) = ay_1(x)\ln(x) + x^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right].$$

  Here, a is the limit as r approaches r_2 of $(r - r_2)a_N(r)$ and $c_n(r_2)$ is the derivative of $(r - r_2)a_n(r)$ with respect to r evaluated at r_2 for $n = 1, 2, \ldots$. These can be easily determined by plugging in y_2, y_2', y_2'' into the original equation since all log factors will disappear. Also, notice that the coefficient c_N can be determined arbitrarily.

Here, $b_n(r_2)$, $c_n(r_2)$, and constant a can also be determined by substituting the form of the series solutions above into the equation (3). A third option is to use reduction of order. In each case, the solutions y_1 and y_2 form a fundamental set of solutions and thus the general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

Though your book doesn't do any examples in this section, you can find several examples in the next section, Section 5.7. In this section, you will see several examples of using these methods to find series solutions for Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

Example: Consider the following equation:

$$xy'' + y = 0.$$

1. Show that $x = 0$ is a regular singular point.

   - Why is it singular?

   - To check it is regular, make sure that the limits in formulas (1) and (2) exist. What are the limits p_0 and q_0 equal to?

2. Find the exponents at the singular point $x = 0$.

   - Use equation (4) to find the indicial equation, $F(r)$.

   - What are the roots?

3. Find the first three nonzero terms of the first solution about $x = 0$.

   - Write the equation above in the correct form (from equation (3)).

   - Guess that a solution looks like $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$. Take derivatives and plug in.

- Bring in the ‘x’ in the first series, and then rewrite so that each series has main term x^{n+r}.

- Check that the coefficient of x^r is $a_0 F(r)$.

- Set the coefficient of x^{n+r} equal to zero for $n \geq 1$ in order to find a recurrence relation that depends on r.

- Plug in $r = r_1$, the larger root, to the recurrence. Use the recurrence to find the first few nonzero terms of y_1. Then y_1 is determined by equation (5).

4. Find the first three nonzero terms of the second solution about $x = 0$.

   - Which case are we in? (Check roots of indicial equation.)

   - What is the correct guess for $y_2(x)$?

   - Take the derivatives and plug in. Why do the log terms disappear?

- Take $c_N = 0$. Find the first few nonzero terms of $y_2(x)$.