Power Series Notes

A **power series** is a series of the form:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Recall the meaning of this notation:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots .$$

Let's go through some of the important ideas about power series we will need for series solutions to differential equations.

1. **Convergence.** A power series is said to **converge** at a point x if the limit of the partial sums exists, that is, the limit

   $$\lim_{m\to\infty} \sum_{n=0}^{m} a_n (x - x_0)$$

   exists for that x. Notice that the series will certainly converge for $x = x_0$. The same power series is said to converge absolutely at a point x if the series

   $$\sum_{n=0}^{\infty} |a_n||x - x_0|^n$$

   converges. Certainly, if a power series converges absolutely, then it also converges. A useful test for absolute convergence of a power series is the **ratio test**. If for some x, consider the limit of the ratio of the $n+1$st term to the nth term:

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0| L.$$

   The power series converges absolutely at a value x if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. The test is inconclusive when $|x - x_0|L = 1$. The **radius of convergence**, ρ, is the number so that a power series converges absolutely for all $|x - x_0| < \rho$. The **interval of convergence** is the interval around x_0 for which $f(x)$ converges. (Remember to check endpoints!)

   Example: Let's find the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n}(x+1)^n$.

   We use the ratio test and set it to less than 1.

   $$\lim_{n\to\infty} \left| \frac{2^{n+1}(x+1)^{n+1} n}{2^n (x+1)^n (n+1)} \right| = 2|x+1| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 2|x+1| < 1.$$

From this, we obtain $|x+1| < 1/2$. Therefore, it follows immediately from the ratio test that the radius of convergence is $1/2$.

Convergence is important for series solutions to differential equations because we need to know that the power series converges at a particular x in order to know that the solution exists at that point x.

2. **Adding/subtracting series.** Series can be added and subtracted just like polynomials (you add and subtract the coefficients). If $f(x)$ and $g(x)$ are power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

then the sum/difference of $f(x)$ and $g(x)$ is

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n.$$

If $f(x) = g(x)$, then $a_n = b_n$ for all n. Also, if $f(x) = 0$, then $a_n = 0$ for all n.

3. **Differentiating series.** Power series can be differentiated term-wise:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}.$$

Question: What is $f''(x)$? $f'''(x)$?

Question: What is $f(x_0)$? $f'(x_0)$? $f''(x_0)$? What about $f^{(n)}(x_0)$ for general n?

The **Taylor series** for a function $f(x)$ about $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

If the radius of convergence for this Taylor series is positive, $\rho > 0$, then we say that the Taylor series is **analytic** at $x = x_0$.

4. **Shift of Index of Summation.** Often, we will want to write a power series with term $(x - x_0)^n$. Suppose we have

$$\sum_{n=a}^{\infty} a_n (x - x_0)^{n-c}.$$

To write this with the term $(x - x_0)^n$, shift the indices to $m = n - c$. This means that $n = m + c$, so plug in $m + c$ wherever you see n, to get:

$$\sum_{m=0}^{\infty} a_{m+c} (x - x_0)^m.$$

Since m is just a dummy variable, we can write this as:

$$\sum_{n=0}^{\infty} a_{n+c} (x - x_0)^n.$$

Question: Rewrite $f'(x)$ (as we found above) with the term $(x - x_0)^n$.

Question: Rewrite $f''(x)$ (as we found above) with the term $(x - x_0)^n$.