Laplace Transform (Optional) Extra Credit Assignment

Please see Chapter 6 for an overview of the Laplace Transform. You are welcome to use other sources as well. You can gain up to 1 point added to your final grade.

1. What is the Laplace transform? Why is it useful for solving differential equations?

2. Find the Laplace transform of
$$f(t) = e^{4t} - \cos(2t) + e^{4t}\cos(2t).$$
Do not compute this from the definition! Use linearity and Table 6.2.1.

3. Find the inverse Laplace transform of
$$F(s) = \frac{2}{s+2} + \frac{8}{3s^2+12} + \frac{9}{s^5}.$$

4. Use the Laplace transform to find the solution to
$$ty'' - ty' + y = 4$$
with $y(0) = 4$ and $y'(0) = 2$.

5. Let $u_c(t)$ be the **Heaviside function**, defined for $c \geq 0$ to be
$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$
The Laplace transform of the Heaviside function is
$$\mathcal{L}(u_c(t)) = \frac{e^{-cs}}{s}, \quad s > 0.$$
Consider the initial value problem $y'' + y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$, where
$$g(t) = u_0 + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t).$$
   (a) Draw the graph of $g(t)$ for $0 \leq t \leq 6\pi$.
   (b) Find the solution to the initial value problem.
   (c) Let $n = 15$ and plot the graph of the solution for $0 \leq t \leq 60$. Describe the solution and explain why it behaves as it does.
   (d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?
   (e) Describe a physical situation which could be described by the above differential equation and explain what the solution means physically.