Example:

$$\vec{x}' = \begin{pmatrix} -5 & 3 \\ -3 & 1 \end{pmatrix} \vec{x}$$

Eigenvalues: $\begin{vmatrix} -5-\lambda & 3 \\ -3 & 1-\lambda \end{vmatrix} = (-5-\lambda)(1-\lambda) + 9$

$= \lambda^2 + 4\lambda + 4 = 0$

$\lambda = -2, -2$, Repeated.

Find an eigenvector:

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow -3\eta_1 + 3\eta_2 = 0$$

$\eta_1 = \eta_2$

so $\vec{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Find generalized eigenvector: Find ρ which satisfies $(A - \lambda I)\vec{\rho} = \vec{\eta}$

$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So $-3\rho_1 + 3\rho_2 = 1$

Let $\rho_1 = 1$, then $\rho_2 = \frac{4}{3}$.

So $\vec{\rho} = \begin{pmatrix} 1 \\ 4/3 \end{pmatrix} \leftarrow$ one poss. generalized eigenvector.

Solution: $\vec{x} = c_1 \vec{\eta} e^{\lambda t} + c_2 \left[\vec{\eta} t e^{\lambda t} + \vec{\rho} e^{\lambda t} \right]$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 4/3 \end{pmatrix} e^{-2t} \right]$$