Practice Problems for Math 23 Final

NOTE: This review only includes new material. However, the final exam is cumulative and may include questions from the first half of class.

Chapter 7:

1. Find the general solution to the following system of differential equations:

$$\boldsymbol{x}' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \boldsymbol{x}$$

Chapter 9:

2. Consider the nonlinear system of equations. Find all critical points and determine their type and stability.

$$\begin{aligned} x' &= x(y-3) \\ y' &= y(x-2). \end{aligned}$$

Chapter 5:

3. For each differential equation, determine whether each of the points $x_0 = 1$, $x_0 = -2$, and $x_0 = 0$ is ordinary or singular. If it is singular, determine whether it is regular.

   (a) $(x+2)^2(x-1)y' + 3(x-1)y'' - 2(x+2)y = 0$

   (b) $(x^2 - x)y'' + y = 0$

4. For each, find the first 3 nonzero solutions to each of the series solutions.

   (a) $(1-x)y'' + xy' + y = 0$ around $x_0 = 0$.

   (b) $x^2y'' + xy' + (2x^2 - 1)y = 0$ around $x_0 = 0$.

5. For part (a) of the previous problem:

   (a) Show that y_1 and y_2 form a fundamental set of solutions.

   (b) Determine a lower bound for the radius of convergence for the series solution.

   (c) Find the solution given that $y(0) = 1$ and $y'(0) = 2$.

Chapter 10:

6. Find the Fourier series for the periodic function $f(x)$ which is defined on the interval $0 \leq x < 1$ to be
$$f(x) = 1 - x,$$
and defined outside of that interval to be $f(x+1) = f(x)$.

7. Describe the method of separation of variables and explain why it is useful.

8. Suppose $u(x,t)$ describes the temperature of a rod of length π at time t for $t \geq 0$ and position x for $0 \leq x \leq \pi$ and satisfies the heat equation
$$4u_{xx} = u_t.$$
If we assume that the endpoints of the rod are kept at temperature 0 and $u(x,0) = 10$ for all $0 < x < \pi$, then find the formula for $u(x,t)$.

9. Suppose $u(x,t)$ describes the displacement of an elastic string of length 4 and satisfies the wave equation
$$u_{xx} = u_{tt}.$$
Assume that the endpoints are kept fixed, and that $u(x,t)$ satisfies initial conditions
$$u(x,0) = x^2 - 4x$$
and
$$u_t(x,0) = 1.$$
Find a formula for $u(x,t)$.