Math 23, Spring 2007

Lecture 9

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Outline

Last class

Today's material

- Complex roots
- Repeated Roots
- Repeated Roots
- Repeated Roots
- Reduction of order

Group work

Next class

Material from last class

- Wronskian: linear independence
- Constant coeffecient equations: complex roots

$$ay'' + by' + cy = 0$$

where $b^2 - 4ac < 0$

- Fundamental set of solutions:

$$y_1(t) = e^{\alpha t}\cos(\beta t)\ y_2(t) = e^{\alpha t}\sin(\beta t)$$

where $\alpha = -\frac{b}{2a}, \beta = \frac{\sqrt{4ac-b^2}}{2a}$

Qualitative behavior

Given a fundamental set of solutions

$$y_1(t) = e^{\alpha t}\cos(\beta t)\ y_2(t) = e^{\alpha t}\sin(\beta t)$$

We make the following observations:

- Solutions of this form are oscillatory
- β small: long periods
- β large: short periods
- $\alpha > 0$: solutions tend to ∞
- $\alpha < 0$: solutions tend to 0
- $\alpha = 0$: solutions are periodic

Qualitative behavior

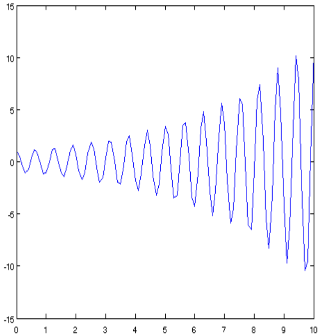

Figure: $\alpha > 0$

Qualitative behavior

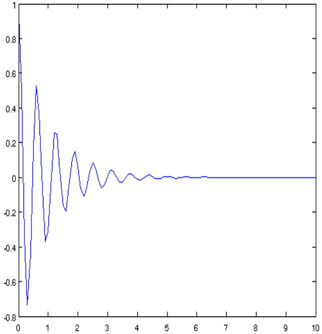

Figure: $\alpha < 0$

Repeated roots

The last case left for constant coeeficient linear homogeneous ODE is the case where we have a double root of the characteristic equation (i.e. $b^2 - 4ac = 0$). In this case, our method produces a single solution

$$y_1(t) = e^{\frac{-b}{2a}t}$$

The theory we have studied requires us to have two linearly independent solutions. How can we find a second solution?

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Reduction of order

We know that, given a single solution, y_1, Cy_1 is also a solution for any constant C.

Idea: look for solutions of the form $y_2(t) = v(t)y_1(t)$.

$$y_2' = v'y_1 + vy_1'$$
$$y_2'' = v''y_1 + 2v'y_1' + vy1''$$

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Reduction of order

Plugging this into the equation

$$a(v''y_1 + 2v'y_1' + vy_1'') + b(v'y_1 + vy_1') + c(vy_1) =$$

$$v(ay_1'' + by_1' + cy_1) + (av''y_1 + 2av'y_1' + bv'y_1)$$

$$= v(0) + av''e^{-\frac{b}{2a}t} + \left(2av'\frac{-b}{2a} + bv'\right)e^{-\frac{b}{2a}t}$$

$$= av''e^{-\frac{-b}{2a}t}$$

Conclusion: $v'' = 0$, i.e $v(t) = c_1t + c_2$.

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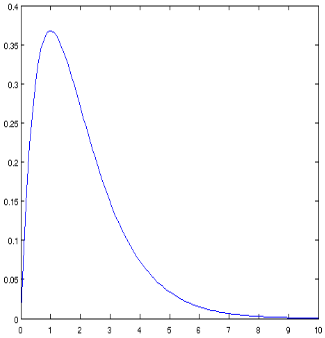

Figure: $y_2(t) = te^{-t}$

Reduction of order

For the case of repeated roots, our method has yielded two solutions:

$$y_1(t) = e^{-\frac{b}{2a}t},\ y_2 = te^{-\frac{b}{2a}t}$$

Do these form a fundamental set of solutions?

$$W(y_1, y_2, t) = det \begin{pmatrix} e^{-\frac{b}{2a}t} & te^{-\frac{b}{2a}t} \\ -\frac{b}{2a}e^{-\frac{b}{2a}t} & e^{-\frac{b}{2a}t} - t\frac{b}{2a}e^{-\frac{b}{2a}t} \end{pmatrix} = e^{-\frac{b}{a}t}$$

Conclusion: these solutions for a fundamental set of solutions

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Conclusion: these solutions for a fundamental set of solutions

General second order linear equations

Reduction of order

If you have one solution, $y_1(t)$, of a differential equation

$$y'' + p(t)y' + q(t)y = 0$$

We can use reduction of order to try to find another by guessing the second solution has the form $y_2(t) = v(t)y_1(t)$.

If we plug this into the ODE, we get an auxillary equation

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

This is a first order equation in v' and can be solved using earlier techniques (e.g. integrating factors).

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1. Solve the following initial value problem

$$y'' - 2y' + y = 0, \; y(0) = 1, y'(0) = 1$$

2. Consider the ODE

$$t^2 y'' - 4ty' + 6y = 0$$

where $t > 0$.

- Confirm that $y_1(t) = t^2$ is a solution
- Find a second solution using reduction of order

Work for next class

- Reading: 3.6, 3.7
- Homework 4 is due monday 4/23