Math 23, Spring 2007

Lecture 7

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4/11/07

Outline

Material from last class

- Linear homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0$$

- Characteristic equation

$$ar^2 + br + c = 0$$

- Linear combinations of solutions

General second order equations

The most general linear second order equation is of the form:

$$y'' + p(t)y' + q(t)y = g(t)$$

We often write this in operator notation:

$$L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$$

Then the ODE becomes

$$L(y(t)) = g(t)$$

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Linear second order equations

Existence and uniqueness

Theorem

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),\ \ y(t_0) = y_0, y'(t_0) = y_0'$$

where p, q and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

Linear homogeneous second order equations

The principle of superposition

Theorem

If y_1 and y_2 are solutions to $L[y] = 0$ then any linear combination of y_1 and y_2 is also a solution.

Q: Will this be enough to find the unique solution if L satisfies the conditions of the existence and uniqueness theorem?

Linear homogeneous second order equations

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Linear homogeneous second order equations

To answer this we rewrite

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

in matrix form:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

We know that there is a unique solution $\{c_1, c_2\}$ if

$$det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0$$

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Linear homogeneous first order equations

The Wronskian

$$W(y_1, y_2, t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)$$

is called the Wronskian.

Theorem

Suppose that y_1 and y_2 are solutions to $L[y] = 0$ and $W(y_1, y_2, t_0) \neq 0$. Then, the initial value problem

$$L[y] = 0, \; y(t_0) = y_0, y'(t_0) = t_0'$$

has a unique solution.

Corollary

If the characteristic equation of a second order linear homogeneous constant coefficient ODE has distinct real roots. The an associated initial value problem of this form has a unique solution.

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Find the Wronskian of the following functions:

1. $$e^{2t},\ e^{-3t/2}$$

2. $$e^t \sin(t),\ e^t$$

3. $$\cos(\theta)^2, 1 + \cos(2\theta)$$

At which points are the Wronksians nonzero (i.e. at which points do these functions form a fundamental set of solutions)?

Work for next class

- Reading: 3.3
- Homework 3 is due monday 4/16