Math 23, Spring 2007
Lecture 4

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Outline

Last class

Today's material

- Numerical methods
- Euler's method

Next class

Material from last class

- Existence and uniqueness
- Exact Equations

Approximate solutions

As we have seen, while the techniques we have developed for finding solutions are powerful, they apply to only a small set of first order equations. So, how do we deal with a general first order equation:

$$y' = f(t, y), \; y(t_0) = y_0$$

If this initial value problem has a unique solution then we can develop approximative methods to try to find the solution.

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Euler's method

There are a wide variety of numerical methods for solving ODE - a complete discussion is beyond the scope fo this course. We will focus on some of the original methods for X reasons:

1. These methods form the basis for more complicated and precise methods
2. These methods are, in general, simple and easier to understand
3. In this course, we are not concerned with efficiency of finding a solution

Euler's method

Main ideas:

1. Use the ODE to create a tangent line approximation to the solution curve.
2. To create a good approximation at each point we iterate this construction.

Euler's method

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Euler's method

Step 1

Approximate the solution at the point $t = t_0$ by the tangent line:

$$y(t) \approx \underbrace{y_0}_{\text{initial condition}} + \underbrace{f(t_0, y_0)}_{\text{initial slope}}(t - t_0)$$

The tangent line approximation is accurate for t_1 close to t_0. So, for $|t_1 - t_0|$ small enough, we have that $y(t_1) \approx y_0 + f(t_0, y_0)(t_1 - t_0)$.

Euler's method
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Euler's method

Step 2

We now repeat Step 1 using the approximate value $y(t_1) = y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$. In other words, create the tangent line approximation:

$$y(t) \approx y_1 + f(t_1, y_1)(t - t_1)$$

Again for $|t_2 - t_1|$ small enough, we have

$$y(t_2) \approx y_1 + f(t_1, y_1)(t_2 - t_1)$$

Euler's method
Step k

We now repeat this as many times as we like. If we have completed $k - 1$ steps and have $y(t_{k-1}) = y_k$, we create the tangent line approximation:

$$y(t) \approx y_{k-1} + f(t_{k-1}, y_{k-1})(t - t_{k-1})$$

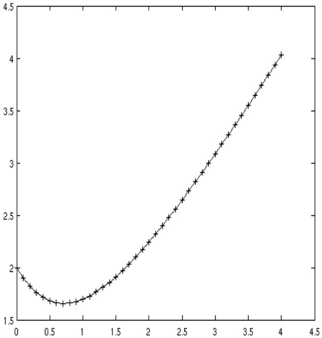
and for $|t_k = t_{k-1}|$ small enough, we have

$$y(t_k) \approx y_{k-1} + f(t_{k-1}, y_{k-1})(t_k - t_{k-1})$$

Euler's method

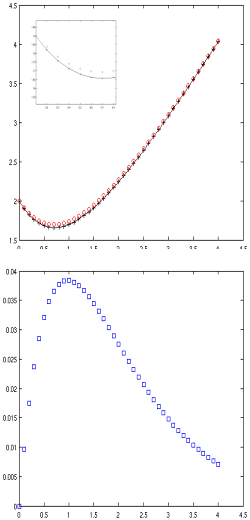
Results

The result of this method is a list of t values $\{t_0, t_1, \ldots, t_N\}$ and corresponding function values $\{y_0, \ldots, y_N\}$. Plotted on a graph, these points give an approximation of the solution curve. Example: $t_{i+1} - t_i = 0.1, N = 40$



Euler's method
Error estimates

How close is the approximation to the actual solution? In our example, we can solve explicitly and compare.



Work for next class

- Reading: 2.3,2.5
- Homework 2 is due 4/9