Math 23, Spring 2007

Lecture 26

Scott Pauls

Department of Mathematics
Dartmouth College

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Material from last class

The heat equation

$$\alpha^2 u_{xx} = u_t$$

1. with conditions $u(x,0) = f(x), u(0,t) = u(L,t) = 0$: Fourier sine series
2. with conditions $u(x,0) = f(x), u_x(0,t) = u_x(L,t) = 0$: Fourier cosine series

The wave equation

$$a^2 u_{xx} = u_{tt}$$

1. with conditions
   $u(x,0) = f(x), u_t(x,0) = g(x), u(0,t) = u(L,t) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L)(a_n \cos(n\pi t/L) + b_n \sin(n\pi t/L))$$

Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

on a region Ω.

Steady state for heat equation.
Heat equation in 2-d: $\alpha^2 \Delta u = u_t$
Wave equation in 2-d: $a^2 \Delta u = u_{tt}$

Boundary conditions:

1. Dirichlet: u takes specified values on the boundary of Ω.
2. Neumann: the normal derivative of u takes specified values on the boundary of Ω.

Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

on a region Ω.

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Rectangular regions

Dirichlet boundary conditions

$$u(x,0) = 0,\ u(x,b) = 0,\ 0 < x < a$$

$$u(0,y) = 0,\ u(a,y) = f(y),\ 0 < y < a$$

Dirichlet boundary conditions

Laplace's equation in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Boundary condition: $u(a, \theta) = f(\theta)$ Separation of variables yields:

$$\Theta'' + \lambda\Theta = 0$$

$$r^2 R'' + rR' - \lambda R = 0$$

Note: Θ must be periodic with period 2π.

$$\lambda > 0 \implies \Theta = c_1 \cos(\sqrt{\lambda}\theta) + c_2 \sin(\sqrt{\lambda}\theta)$$

and $\sqrt{\lambda}$ must be a positive integer.

$$\lambda = 0 \implies \Theta = c_1$$

Dirichlet boundary conditions

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$$\lambda = 0 \implies \Theta = c_1$$

Euler equations

Solution to

$$L(R) = x^2 R'' + xR' - \lambda R = 0$$

Use series solutions!

Problem: $x = 0$ is not an ordinary point.
Alternative method: guess $R = r^s$

$$s = \pm \frac{\sqrt{4\lambda}}{2}$$

If $\lambda > 0$ then

$$R(r) = C_1 r^{\sqrt{\lambda}} + C_2 r^{-\sqrt{\lambda}}$$

If $\lambda = 0$ we get (using reduction of order for the second solution):

$$R(r) = C_1 + C_2 \ln r$$

Euler equations

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Putting this together

Most general solution so far:

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n(c_n \sin(n\theta) + k_n \cos(n\theta))$$

At $r = a$,

$$u(a,\theta) = f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n(c_n \sin(n\theta) + k_n \cos(n\theta))$$

So,

$$c_n = \frac{1}{a^n\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\ d\theta$$

$$k_n = \frac{1}{a^n\pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\ d\theta$$

Work for next class

- Read 10.5,10.7,10.8
- Homework 9 assigned but is not due! These are practice problems for the final.