Math 23, Spring 2007

Lecture 25

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5/23/07

Material from last class

The heat equation

$$\alpha^2 u_{xx} = u_t$$

1. with conditions $u(x,0) = f(x), u(0,t) = u(L,t) = 0$: Fourier sine series
2. with conditions $u(x,0) = f(x), u_x(0,t) = u_x(L,t) = 0$: Fourier cosine series

Wave equation

A model for wave propagation in one-dimensional media:

$$a^2 u_{xx} = u_{tt}$$

Initial conditions: $u(x, 0) = f(x), u_t(x, 0) = g(x)$
Boundary conditions: Fixed ends - $u(0, t) = u(L, t) = 0$

Wave equation

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Separation of variables

First case: $g(x) = 0$

Separation of variables yields:

$$X'' - \lambda X = 0, \quad T'' - \lambda a^2 T = 0$$

$$X(0) = 0 = X(L), T'(0) = 0$$

$X(x) = \sin(n\pi x/L), T(t) = \cos(n\pi at/L)$

$$u_n(x,t) = \sin(n\pi x/L)\cos(n\pi at/L)$$

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Superposition

Most general solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)\cos(n\pi a t/L)$$

At $t = 0$,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

So, expand f as a Fourier sine series

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$$

Separation of variables

Second case: $f(x) = 0$

Separation of variables yields:

$$X'' - \lambda X = 0, \quad T'' - \lambda a^2 T = 0$$

$$X(0) = 0 = X(L), T(0) = 0$$

$X(x) = \sin(n\pi x/L), T(t) = \sin(n\pi at/L)$

$$u_n(x,t) = \sin(n\pi x/L)\sin(n\pi at/L)$$

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Superposition

Most general solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)\sin(n\pi at/L)$$

At $t = 0$,

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi a c_n}{L} \sin(n\pi x/L)$$

So, expand f as a Fourier sine series

$$\frac{n\pi a c_n}{L} = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$$

Separation of variables

Third case: General $f(x), g(x)$

Simply add together the two previous solutions

Applet

http://falstad.com/loadedstring/

Work for next class

- Read 10.5,10.7,10.8
- Homework 9 assigned but is not due! These are practice problems for the final.