Math 23, Spring 2007

Lecture 23

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Material from last class

- Fourier series
- Computing Fourier coefficients
- A computation from last time:

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ x, & 0 \le x \le 1 \end{cases}$$

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x) \sin(n\pi x)\, dx \\ &= -\int_{-1}^{0} x \sin(n\pi x)\, dx + \int_{0}^{1} x \sin(n\pi x)\, dx \\ &= \frac{1}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\cos(n\pi) = 0 \end{aligned}$$

Visualizations

A nice applet for looking at Fourier series:
`http://falstad.com/fourier/`

Observations:

- Discontinuities in f lead to convergence problems in the series: the Gibbs effect
- Some functions have only sin terms while others have only cos terms.

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Even and Odd functions

Definition

A function f is even if $f(-x) = f(x)$ while a function is odd if $f(-x) = -f(x)$

Examples: $\sin(x)$ is odd, $\cos(x)$ is even

Even and Odd functions

Properties:

1. The sum (resp. difference) and product (resp. quotient) of two even functions are even.
2. The sum (resp. difference) of two odd functions is odd. The product (resp. quotient) of two odd functions is even.
3. The product (resp. quotient) of an odd function and an even function is odd.
4. If f is odd then
$$\int_{-L}^{L} f(x)\, dx = 0$$
5. If f is even then
$$\int_{-L}^{L} f(x)\, dx = 2\int_{0}^{L} f(x)\, dx$$

Fourier cosine series

Suppose that f, f' are piecewise continuous on $-L \leq x < L$ and that f is an even periodic function with period $2L$. Then,

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = 0$$

f is said to have a Fourier cosine series.

Fourier sine series

Suppose that f, f' are piecewise continuous on $-L \leq x < L$ and that f is an odd periodic function with period $2L$. Then,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$a_n = 0$$

f is said to have a Fourier sine series.

Half range expansions

We can extend our analysis to function defined on an interval of the form $0 \leq x < L$.

1. Extend f as an even or odd function on $[-L, L)$
2. Extend the extension to be periodic
3. Compute Fourier cosine or sine series

Heat flow

A model for heat conduction in a thin heat conducting solid bar. Let $u(x, t)$ denote the temperature at point x on the bar at time t. Then,

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < L, t > 0$$

α^2 is a constant known as the thermal diffusivity.

Additional assumptions:

- An initial temperature distribution: $u(x, 0) = f(x)$
- Boundary conditions on the ends of the rod. For example:
  1. Fixed temperature: $u(0, t) = a, u(L, t) = b$
  2. Insulated ends: $u_x(0, t) = 0 = u_x(L, t)$

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Work for next class

- Read 10.5,10.7,10.8
- Homework 8 is due Monday 5/21/07