Math 23, Spring 2007

Lecture 21

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Material from last class

- First order Linear Systems of equations with constant coefficients

$$\vec{x}' = A\vec{x}$$

- Reviewed qualitative results
- Critical points of nonlinear systems and their classification

IVPs

Up to this point, we have always dealt with initial value problems: ODE(s) with values specified at a single point. E.g.

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, y'(t_0) = y_0'$$

We could also specify data at two different points:

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, y(t_1) = y_1$$

This is called a **two point boundary value problem.**

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Existence and Uniqueness

Recall that for an initial value problem, we often had existence and uniqueness. E.g. for linear second order equations, if p, q, g are continuous on an open interval containing t_0, we have existence of a unique solution.

Do we have similar results for two-point boundary value problems?

Examples

$$y'' + y = 0, \quad y(0) = 1, y(\pi) = -1$$

$r = \pm i, \quad y(t) = c_1 \cos(t) + c_2 \sin(t)$

$$y(0) = 1 \implies c_1 = 1$$

$$y(\pi) = -1 \quad \text{is satisfied}$$

Unique solution!

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$$y'' + y = 0, \quad y(0) = 0, y(\pi) = 0$$

$r = \pm i, \quad y(t) = c_1 \cos(t) + c_2 \sin(t)$

$$y(0) = 0 \implies c_1 = 0$$

$$y(\pi) = 0$$

Infinitely many solutions!

Examples

$$y'' + y = 0, \quad y(0) = 0, y(\pi) = 0$$

$$r = \pm i, \quad y(t) = c_1 \cos(t) + c_2 \sin(t)$$

$$y(0) = 0 \implies c_1 = 0$$

$$y(\pi) = 0$$

Infinitely many solutions!

Examples

$$y'' - 3y' + 2 = 0, \; y(0) = 0, y(1) = 0$$

$r = 1, 2, \quad y(t) = c_1 e^t + c_2 e^{2t}$

$$y(0) = 0 \implies c_1 = -c_2$$

$$y(1) = 0 \implies c_1(e - e^2) = 0 \implies c_1 = 0$$

Only the trivial solution!

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Only the trivial solution!

Conclusion

For two point boundary value problems we may have no solution, a unique solution, or infinitely many solutions.

e.g.

$$y'' + \lambda^2 y = 0$$

$$y(0) = 0, y(\pi) = 0$$

Values of λ where nontrivial solutions exist are called **eigenvalues** and the associated solutions are called **eigenfunctions.**

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Examples

Find the eigenvalues and eigenfunctions for

$$y'' - \mu^2 y = 0$$

$$y(0) = 0, y(\pi) = 0$$

and

$$y'' = 0$$

$$y(0) = 0, y(\pi) = 1$$

Work for next class

- Read 10.2-10.3
- Homework 8 is due Monday 5/21/07