Math 23, Spring 2007

Lecture 19

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Material from last class

- First order Linear Systems of equations with constant coefficients

$$\vec{x}' = A\vec{x}$$

- Method of solution: guess $\vec{x} = \xi e^{rt}$ which relates solutions to the eigenvalues and eigenvectors of A
- Distinct real roots: exponential solutions
- Complex roots: exponentials/periodic solutions

Repeated roots

$$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$$

Eigenvalue: $\lambda = 2$
Eigenvector: $\vec{\xi} = (1, -1)^t$

This gives a single solution $\vec{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$.

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What to do?

In the second order constant coefficient case, we used reduction of order and found solutions of the form $y = te^{rt}$. So, we will try to find additional solutions of the form

$$\vec{x} = \xi te^{rt} + \eta e^{2t}$$

Plugging this into our equation yields

$$\xi e^{2t} + 2\xi te^{2t} + 2\eta e^{2t} = A\xi te^{2t} + A\eta e^{2t}$$

Or, cancelling the e^{2t},

$$(\xi + 2\eta) + 2t\xi = A\eta + A\xi t$$

Equating the coefficients yields

$$(A - 2I)\xi = 0$$

and

$$(A - 2I)\eta = \xi$$

This first equation is satisfied if ξ is an eigenvector associated to $\lambda = 2$ so we are left with solving the second equation for η.

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Example (con't)

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solve for $\eta = (k, -1-k)^t = (0, -1)^t + k(1, -1)^t$
So our two solutions are

$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

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$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

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Equilibrium solution

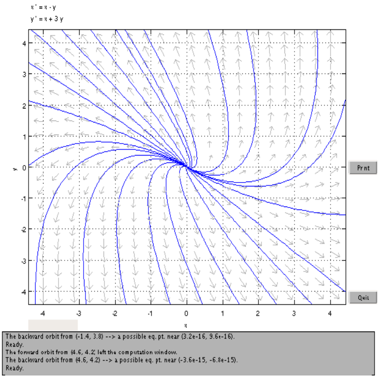

Figure: An improper node

As $t \to \infty$, these solutions tend to infinity along a line of slope -1. Thus it is similar to a node. This case is called an **improper node.** Our example is asymptotically unstable but if the eigenvalue were negative, it would be asymptotically stable.

Example

$$\vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$$

1. Find eigenvalues and eigenvectors (you should have one repeated eigenvalue)
2. Find a second solution using the method we learned today
3. Classify the equilibrium point at $x = 0$.

Work for next class

- Read 9.1-9.3
- Homework 7 is due Monday 5/14/07