# Math 23, Spring 2007 

Lecture 18

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Dartmouth College

5/7/07

## Material from last class

- First order Linear Systems of equations with constant coefficients

$$
\vec{x}^{\prime}=A \vec{x}
$$

- Method of solution: guess $\vec{x}=\xi e^{r t}$ which relates solutions to the eigenvalues and eigenvectors of $A$


## Equilibrium points

For

$$
\vec{x}=A \vec{x}
$$

we have any $\vec{x}$ with $A \vec{x}=0$ as an equilibrium point. In particular, $\vec{x}=0$ is always such a point.

Classification: let $\lambda_{1}, \lambda_{2}$ be the eigenvalues of $A$

1. $\lambda_{1}<0<\lambda_{2}$, real: zero is a saddle point and is an unstable equilibrium point.
2. $\lambda_{1}, \lambda_{2}$ real, nonzero and of the same sign: zero is a node is asymptotically stable, if $\lambda_{i}<0$, and asympotically unstable otherwise

## Equilibrium points

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## Equilibrium points



Today's material
Solution for linear first order systems

Figure: A saddle point

## Equilibrium points

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## Last class



The backard orb it tron B.5, -3.2 lef the conpulat on wind dow.
R.2dy.

R.zady.

Figure: A node (stable)

## Complex eigenvalues

As we have seen, we may have two complex conjugate eigenvalues:

$$
\vec{x}^{\prime}=\left(\begin{array}{ll}
3 & -2 \\
4 & -1
\end{array}\right) \vec{x}
$$

Eigenvalues: $1 \pm 2 i$

## Complex eigenvalues

As we have seen, we may have two complex conjugate eigenvalues:

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\vec{x}^{\prime}=\left(\begin{array}{ll}
3 & -2 \\
4 & -1
\end{array}\right) \vec{x}
$$

Eigenvalues: $1 \pm 2 i$

## Solution:

$$
\vec{x}=\xi e^{(1 \pm 2 i) t}
$$

## Complex eigenvalues

## Recall Euler's equation:

$$
e^{i t}=\cos (t)+i \sin (t)
$$

Last class
Today's material
Solution for linear first order systems

Suppose we have eigenvalues $\alpha \pm i \beta$ and eigenvectors $a \pm i b$. Then
$\vec{x}=(a+i b) e^{(a+i \beta) t}$
$=e^{\alpha t}((a \cos (\beta t)-b \sin (\beta t))+i(a \sin (\beta t)+b \cos (\beta t)))$

## and

$(a-i b) e^{(\alpha-i \beta) t}$
$=e^{\alpha t}((a \cos (\beta t)+b \sin (\beta t))+i(-a \sin (\beta t)-b \cos (\beta t)))$

## Complex eigenvalues

Recall Euler's equation:

$$
e^{i t}=\cos (t)+i \sin (t)
$$

Suppose we have eigenvalues $\alpha \pm i \beta$ and eigenvectors $a \pm i b$. Then

$$
\begin{aligned}
\vec{x} & =(a+i b) e^{(\alpha+i \beta) t} \\
& =e^{\alpha t}((a \cos (\beta t)-b \sin (\beta t))+i(a \sin (\beta t)+b \cos (\beta t)))
\end{aligned}
$$

and

$$
\begin{aligned}
& (a-i b) e^{(\alpha-i \beta) t} \\
& \quad=e^{\alpha t}((a \cos (\beta t)+b \sin (\beta t))+i(-a \sin (\beta t)-b \cos (\beta t)))
\end{aligned}
$$

## Complex eigenvalues

Taking an appropriate linear combination yields two real functions

$$
\begin{aligned}
& u(t)=e^{\alpha t}((a \cos (\beta t)-b \sin (\beta t)) \\
& v(t)=e^{\alpha t}((a \sin (\beta t)+b \cos (\beta t))
\end{aligned}
$$

Caution: remember that $a$ and $b$ are vectors.

## Example

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Last class
Today's material
Solution for linear first order systems

$$
\vec{x}^{\prime}=\left(\begin{array}{ll}
3 & -2 \\
4 & -1
\end{array}\right) \vec{x}
$$

## Classification of equilibrium points

1. $\lambda=a \pm i b$ : zero is a spiral point and is asympototically stable if $a<0$ and unstable otherwise.
2. $\lambda-a \pm i b, a=0$ : zero si called a center and is stable but not asymptotically stable.

## Equilibrium points

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## Last class




The buckard orbit from $(-2.8,-2.8)$ left the conpulat in windom
R:zady.

Today's material
Solution for linear first order systems

Next class

Figure: A spiral point: asympt. stable

## Equilibrium points



Today's material
Solution for linear first order systems

Figure: A sprial point:asympt. unstable

## Equilibrium points



Today's material
Solution for linear first order systems

Figure: A center

## Example

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Last class
Today's material
Solution for linear first order systems

$$
\vec{x}^{\prime}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
2 & 1 & -2 \\
3 & 2 & 1
\end{array}\right) \vec{x}
$$

## Work for next class

- Read 7.7
- Homework 7 is due Monday 5/14/07

