Math 23, Spring 2007
Lecture 12

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Outline

Material from last class

- Spring-mass systems

$$mu'' + \gamma u' + ku = 0$$

- No damping: periodic solutions
- Damping: three cases - underdamping, critical damping and overdamping

Forcing with damping

$$mu'' + \gamma u' + ku = F_e(t)$$

As an example, we will consider $F_e(t) = F_0 \cos(\omega t)$ for some constants F_0, ω.

Recall the general solution in this case will be

$$u(t) = u_c(t) + U(t) = \text{"homog. sol."} + \text{"particular sol."}$$

As we saw last class, the roots of the characteristic equation for the homogeneous equation must be negative which says that

$$\lim_{t \to \infty} u_c(t) = 0$$

$u_c(t)$ is called the transient solution while $U(t)$ is the steady state solution or the forced response.

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Forcing with damping

Using the method of undetermined coefficients and some trig identities, we can show that

$$U(t) = R\cos(\omega t - \delta)$$

where

$$R = \frac{F_0}{\Delta}, \ \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \ \sin\delta = \frac{\gamma\omega}{\Delta}$$

and

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \ \omega_0^2 = \frac{k}{m}$$

R is called the *amplitude* and δ is called the *phase* of the solution.

Resonance

When is R the largest?

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

A: when ω_0 is close to ω

In this case, the (relatively) small external force has a very large impact. To further investigate this, we can compute

$$R_{max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - \frac{\gamma^2}{4mk}}}$$

which occurs when

$$\omega_{max} = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 = \frac{\gamma^2}{2mk}\right)$$

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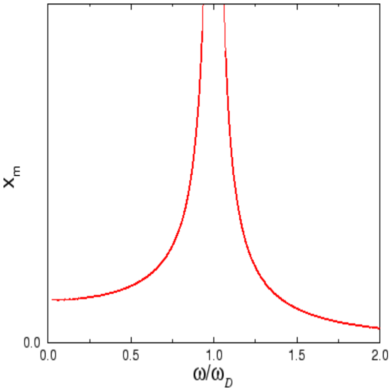

Figure: max R vs. ω/ω_0

See http://www.walter-fendt.de/ph14e/resonance.htm and http://hcgl.eng.ohio-state.edu/ ce406/Chapt6/Tacoma1.mpg

Forcing with no damping

$$mu'' + ku = F_0 \cos(\omega t)$$

With $\omega_0^2 = k/m$. If $\omega \neq \omega_0$, the general solution is

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Under initial condition $u(0) = 0, u'(0) = 0$, we have

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, c_2 = 0$$

Simplification yields

$$u = \left(\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)t}{2} \right) \right) \sin\left(\frac{(\omega_0 + \omega)t}{2} \right)$$

Interpretation: This is a sinusoidal function with varying (again sinusoidal) amplitude.

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Work for next class

- Exam tomorrow
- Homework 5 is due wednesday 5/1
- Midterm exam: Next tuesday. Covers through section 3.7
- See webpage for review materials