Math 23, Spring 2007

Lecture 10

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4/18/07

Outline

Last class

Today's material

- Inhomogeneous equations
- Particular solutions
  - Method of undetermined coefficients
- Variation of parameters

Next class

Material from last class

- Constant coeffecient equations

$$ay'' + by' + cy = 0$$

Three cases:

1. $b^2 - 4ac > 0$: distinct real roots
   Fundamental set of solutions:

$$y_1(t) = e^{r_1 t},\ y_2(t) = e^{r_2 t}$$

2. $b^2 - 4ac = 0$: double real root
   Fundamental set of solutions:

$$y_1(t) = e^{r_1 t},\ y_2(t) = te^{r_1 t}$$

3. $b^2 - 4ac < 0$: complex roots
   Fundamental set of solutions:

$$y_1(t) = e^{\alpha t}\cos(\beta t)\ y_2(t) = e^{\alpha t}\sin(\beta t)$$

- Reduction of order

Inhomogeneous equations

So far, we have focused on homogeneous equations

$$y'' + p(t)y' + q(t)y = 0 \quad (1)$$

We now turn to inhomogeneous equations

$$y'' + p(t)y' + q(t)y = g(t) \quad (2)$$

Observations:

- If we can find a single solution $y_p(t)$ to (2) then we can add on the general solution to (1) to create a two parameter family of solutions:

$$y_p(t) + C_1 y_1(t) + C_2 y_2(t)$$

- The set $\{y_p(t) + C_1 y_1(t), y_p(t) + C_2 y_2(t)\}$ is a fundamental set of solutions if $\{y_p, y_1, y_2\}$ are pairwise linearly independent.
- Main goal: find a particular solution

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Method of undetermined coefficients

$$ay'' + by' + cy = g(t)$$

Basic idea: Guess the most general solution that looks like $g(t)$

Examples:

- If g is a polynomial, guess that is a polynomial with unspecified coefficients
- If g is an exponential, guess a similar exponential
- if g contains trigonometric functions, guess a similar combination of trigonmetric functions

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Examples

$$2y'' + 3y' + y = t^2$$

Guess: $t^s(a_0 + a_1 t + a_2 t^2)$

$$y'' - 2y' - 3y = -3te^{-t}$$

Guess: $At^k e^{-t}$

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Variation of parameters

Given a linear inhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

suppose we have the general solution $c_1 y_1(t) + c_2 y_2(t)$ to the *homogeneous* version of this equation. The main idea is similar to reduction of order: replace the constants with functions of t. i.e. look for solutions of the form

$$u_1(t)y_1(t) + u_2(t)y_2(t)$$

Example:

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Fundamental set of soutions to homogeneous equation: $\{e^{3t}, e^{-t}\}$

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Variation of parameters

Theorem

If the functions p, q and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \tag{3}$$

associated to the inhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{4}$$

then a particular solution of (4) is

$$y_p(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2, s)}\, ds + y_2(s)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2, s)}\, ds$$

where t_0 is any point in I.

Work for next class

- Reading: 3.8, 3.9
- Homework 4 is due monday 4/23
- Midterm exam: Next tuesday. Covers through today's class (section 3.7)