Name:

Question you don't want graded:

Math 23 Diff Eq: Final (Fall 2007)

3 hours, each question worth 15 points. Answer 8 out of 9 questions. You MUST indicate which one you don't want graded above ("whichever is lowest" is not acceptable!). 1 side of notes allowed. No calculator needed; no algebra-capable ones allowed. Read questions carefully so you don't miss parts. Good luck and enjoy!

1. Consider $y'' + \pi^2 y = 0$

   (a) [3 points] Find the general solution $y(x)$.

   (b) [4 points] If $y(0) = 1$ and $y(2) = 1$, find and sketch a solution or explain why there is none. Is it unique?

   (c) [4 points] If $y(0) = 0$ and $y'(1) = 0$ find and sketch a solution or explain why there is none. Is it unique?

(d) [4 points] Find an expression for all eigenvalues of $y'' + \lambda y = 0$ with the above boundary conditions $y(0) = 0$ and $y'(1) = 0$. Be careful to state where your series starts. Sketch the lowest couple of eigenfunctions.

(e) [BONUS 3 points] Prove that there are no *negative* eigenvalues

2. Heat flows in a rod of length 1 according to $u_t = u_{xx}$. It is initially at a uniform temperature 10 deg F. At $t = 0$ the left end is attached to a constant-temperature refrigerator at 0 deg F and the right end to a pile of snow at a constant 20 deg F.

   (a) [3 points] What temperature distribution is approached in the limit $t \to \infty$?

   (b) [12 points] Find the temperature evolution $u(x, t)$ for all $t > 0$. [Hint: use the symmetry of the required temperature change function to check expected terms disappear!]

3. Consider the equations $dx/dt = -x - 3y$ and $dy/dt = 3x - y$.

(a) [7 points] Find the general solution (using real-valued functions).

(b) [3 points] Find the solution with initial condition $x(0) = 1$, $y(0) = 0$.

(c) [2 points] Sketch this motion in the (x, y) plane, being sure to show the direction of motion.

(d) [2 points] Characterize (describe) the type of fixed point. Is the motion stable? Is it asymptotically stable?

(e) [2 points] What is the largest interval of time t over which the solution *must* be unique? By changing one of the equations, show an example system which might not be unique on this interval.

4. Consider the matrix $A = \begin{bmatrix} -1 & -2 & 1 \\ 0 & -4 & 3 \\ 0 & -6 & 5 \end{bmatrix}$.

(a) [4 points] Show that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ are eigenvectors of A, and find the corresponding eigenvalues.

(b) [5 points] Find as many linearly independent eigenvectors as possible.

(c) [6 points] Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$.

5. The Schrödinger equation in a harmonic well, namely,

$$-\frac{1}{2}y'' + 2x^2 y = y$$

is very important in quantum physics. Solve this equation to write a general formula for $y(x)$ using a power series expansion about $x_0 = 0$. (Please give only the first 3 terms of each function in the general solution)

[BONUS 3 point]: Continue the even terms until you spot the pattern, then identify the function it represents.

6. Consider the equation $e^y - 2x + xe^y \frac{dy}{dx} = 0$ with $y(2) = 0$.

   (a) [3 points] What can you say rigorously about existence and uniqueness of the solution (without solving)?

   (b) [10 points] Find an *explicit* formula for the solution $y(x)$.

   (c) [2 points] What is the largest x domain in which your solution is in fact valid?

7. A mass $m = 1$ is attached to a spring constant $k = 25$ and friction causes damping $\gamma = 2$.

   (a) [2 points] Is the system over-, under-, or critically damped?

   (b) [3 points] What is the quasifrequency, in *cycles per second*, and the Q factor?

   (c) [5 points] If a driving force of $\sin 5t$ is applied, find the steady state solution and the general motion.

   (d) [5 points] If damping γ is changed to zero, repeat the previous question.

   (e) [BONUS 2 points] If, with no driving, the same mass now has motion $e^{-t} \sin 100t$, find the new k and γ.

8. The displacement of a violin string in the interval $0 < x < \pi$ satisfies $u_{tt} = c^2 u_{xx}$ where c is the wave speed. The ends of the string are fixed $u(0,t) = u(\pi,t) = 0$, $t > 0$. The string is plucked with the initial displacement $u(x,0) = f(x)$, and zero initial velocity $u_t(x,0) = 0$.

   (a) [10 points] Find an expression for the subsequent motion of the string, $u(x,t)$ for $t > 0$, in terms of the general function f. You may include coefficients but give the formula for them.

   (b) [5 points] In a more realistic model with air friction the string obeys $u_{tt} + 2\alpha u_t = c^2 u_{xx}$, with damping coefficient $\alpha < c$. Use separation of variables to find a formula for the general solution $u(x,t)$ (don't bother matching to any particular initial conditions).

9. Short answer questions

(a) [5 points] Let $f(x) = 1 - x$ on the interval $(0, 1)$. To what number does the Fourier sine series converge to at $x = 3/2$? At $x = 0$?

(b) [3 points] A damped oscillator only has significant response when driven at within 0.01% of its natural frequency. Estimate the number of cycles which occur in one decay time, when the undriven oscillator is released with nonzero initial conditions.

(c) [4 points] State as much as you can (*e.g.* functional form, domain) about the Wronskian of any two linearly-independent solutions to $x^2y'' + xy' + y = 0$. (Explain your answer)

(d) [3 points] Sketch the phase line, labeling a few stable and unstable fixed point locations, for $y' - \sin y = 0$.