Final Exam

MATH 23 — WINTER 2014

NAME:

SECTION: **11** **12**

This exam has 11 questions on 16 pages, for a total of 250 points.

You have 180 minutes to answer all questions.

This is a closed book exam.

Use of calculators and other electronic devices is not permitted.

Show all your work, justify all your answers.

Question	Points	Score
1	20	
2	25	
3	20	
4	35	
5	20	
6	20	
7	20	
8	30	
9	20	
10	20	
11	20	
Total:	250	

20 1. Find the general solution to the differential equation

$$2y' + y = 3t$$

25 2. Solve the following boundary value problem.

$$x^2y'' - 2xy' + 2y = 0, \quad y(1) = -1, \quad y(2) = 1$$

20 3. (a) Find the Fourier series for $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$.

Reduce your answer as much as possible.

(b) Graph the function to which the series converges for three periods.

35 4. Consider a rod of length 20 cm. that is initially at the uniform temperature of 25° C. Suppose that at time 0 the end $x = 0$ is cooled to 0°C. and the end $x = 20$ is heated to 60°C. and both remain at those temperatures thereafter. Further suppose that the rod is made of a material so that $\alpha^2 = 1.5\, cm^2/s$.

(a) Set up the heat conduction problem, that is, state the differential equation, initial conditions, and boundary conditions.

(b) Find the steady state solution for this problem.

(c) Find the temperature distribution in the rod at any time t.
Reduce your answer as much as possible.

20 5. Find the solution to the initial value problem

$$(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0, \quad y(-2) = 2$$

You may leave your answer in implicit form.

20 6. Find the general solution to the differential equation

$$y'' + 2y' = 4\sin(2t)$$

20 7. Consider the wave equation problem

$$\begin{aligned} 9u_{xx} &= u_{tt} \\ u(0,t) = u(4,t) &= 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \begin{cases} 3x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \end{cases} \end{aligned}$$

Find the form of a series solution for $u(x,t)$ and give an integral formula for the coefficients of the series.

Leave your formula in integral form, do not attempt to solve for the coefficients.

30 8. Find the general solution to $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}$ and determine for which values of a the solution with initial value $\vec{x}(0) = \begin{bmatrix} 1 \\ a \end{bmatrix}$ does $x_1(t) \to +\infty$ as $t \to +\infty$.

20 9. For each fundamental set of solutions $\{\vec{x}_1, \vec{x}_2\}$ to a system $\vec{x}' = A\vec{x}$ below, sketch in a phase plane (i.) the solution curves $\vec{x}_1, \vec{x}_2$; (ii.) the solution to the initial value problem

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

(a) $\vec{x}_1 = \begin{bmatrix} e^{5t} \\ -3e^{5t} \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 4e^{t} \\ e^{t} \end{bmatrix}$

(b) $\vec{x}_1 = \begin{bmatrix} e^{t} \\ 3e^{t} \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$

20 10. Let $x_1(t)$ be the population of Gwalia and $x_2(t)$ be the population of Albion at time t, measured in years. Each year in both countries 2% of the population has a baby, and 1.5% of the population dies. In addition, 5% of the population of Gwalia emigrates to Albion, and 10% of the population of Albion emigrates to Gwalia. There is no other immigration or emigration.

Write a system of two differential equations for this situation, and express it as a matrix equation.

20 11. The system of differential equations $\vec{x}' = A\vec{x}$ has fundamental matrix

$$X(t) = \begin{pmatrix} 3e^t & (3t-1)e^t \\ e^t & (t+3)e^t \end{pmatrix}$$

Calculate e^{At}.

Scratch work. Refer to this on the question's page if you want it to be graded.

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