Math 23 Fall 2014

Differential Equations

Final Exam

Friday, November 21, 3:00PM - 6:00PM

Your name (please print): ____

Section (circle one): Olivia Prosper, Min Hyung Cho

Instructions: This is a closed book, closed notes exam. The use of calculators is not permitted. The exam consists of **9** problems and this booklet contains **16** pages (including this one). **On problems 2 through 9, you must show your work and justify your assertions to receive full credit**. Justify your answers and simplify your results as much as possible. Also, please clearly mark your final (simplified) answer. The last two pages of this booklet are blank. Good Luck!

The Honor Principle requires that you neither give nor receive any aid on this exam.

Math 23 Fall 2014

Your name (please print): ______________________________

Problem	Points	Score
1	10	
2	6	
3	10	
4	12	
5	10	
6	20	
7	11	
8	10	
9	11	
Total	**100**	

Work will not be graded in this page

1. Short Answer Questions ($(a) \sim (b)$)-Work will not be graded

   (a) **(5 points)** Consider a uniform rod length π for which $\alpha^2 = 1$. Then, the temperature of the rod follows

   $$u_t = u_{xx}, \qquad 0 < x < \pi.$$

   Suppose initial temperature is given by $u(x,0) = \sin x + \cos x$ and both ends of rod are insulated. Find the steady state temperature in the bar.

   The steady state temperature is c_0 or $c_0/2$.

   $$c_0 = \frac{1}{\pi}\int_0^{\pi} \sin x + \cos x dx = \frac{2}{\pi}$$

   Answer: $\frac{2}{\pi}$

   (b) **(5 points)** Determine the lower bound for the radius of convergence of the power series solution near $x_0 = 1$ of the differential equation

   $$(x^2+4)(x+2)y'' + (x-2)y' - x^2 y = 0.$$

   Answer: $\rho = \sqrt{5}$

Show your work

2. (**6 points**) Determine whether the following set of vectors is linearly independent or linearly dependent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 5 \\ -1 \\ 2 \end{pmatrix}.$$

$$1 \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 5 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Answer: Linearly dependent

Show your work

3. (**10 points)** Find an explicit solution of the initial value problem.

$$\frac{dP}{dt} = (1-P)P, P(0) = \frac{1}{2}$$

Separable equation

$$\frac{1}{(1-P)P}dP = dt$$

By partial fraction

$$(\frac{1}{(1-P)} + \frac{1}{P})dP = dt$$

Integrating both sides

$$-\ln(1-P) + \ln P = t + C \to \ln\frac{P}{1-P} = t + C \to \frac{P}{1-P} = Ce^t$$

$$\to (1-P)Ce^t = P \to P(1+Ce^t) = Ce^t \to P(t) = \frac{Ce^t}{1+Ce^t}$$

Initial condition

$$P(0) = \frac{C}{1+C} = \frac{1}{2} \to C = 1$$

Therefore

$$P(t) = \frac{e^t}{1+e^t}$$

4. **(12 points)** Determine if $f(x) = e^{-x}$ and $g(x) = -x+1$ are orthogonal on $[0, \infty)$ and justify.

Inner product or Dot product

$$\int_0^\infty e^{-x}(1-x)dx = \lim_{t\to\infty} \int_0^t e^{-x}(1-x)dx$$

$$\int_0^t e^{-x}(1-x)dx = \int_0^t e^{-x}dx - \int_0^t xe^{-x}dx$$

$$= [-e^{-x}]_0^t - ([-xe^{-x}]_0^t + \int_0^t e^{-x}dx)$$

$$= -e^{-t} + 1 - (-te^{-t} + [-e^{-x}]_0^t) = -e^{-t} + 1 - (-te^{-t} + (-e^{-t} + 1)) = te^{-t}$$

By L'Hospital Rule

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0$$

Therefore

$$\int_0^\infty e^{-x}(1-x)dx = \lim_{t\to\infty} \int_0^t e^{-x}(1-x)dx = \lim_{t\to\infty} te^{-t} = 0$$

Answer: $f(x)$ and $g(x)$ are orthogonal

5. (**10 points**) Seek a power series solution (Find a recurrence relation) of

$$y^{(3)} + y' = 0$$

near $x_0 = 0$ by assuming a_0, a_1, and a_2 are arbitrary constants.
Let

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, y^{(3)} = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}.$$

Substituting y' and $y^{(3)}$ into differential equation gives

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

By matching the index

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 0$$

$$\sum_{n=0}^{\infty} ((n+3)(n+2)(n+1) a_{n+3} + (n+1) a_{n+1}) x^n = 0$$

Therefore

$$(n+3)(n+2)(n+1) a_{n+3} + (n+1) a_{n+1} = 0 \to a_{n+3} = -\frac{1}{(n+3)(n+2)} a_{n+1}, n = 0, 1, 2, \ldots$$

Show your work

6. An engineer attempts to take damping into account in the vibrating string problem by using the following partial differential equation and boundary conditions.

$$\begin{cases} u_{tt} = u_{xx} - 2u_t & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0 & t \geq 0 \\ u_t(x,0) = \sin 2x & 0 \leq x \leq \pi \\ u(x,0) = 0 & 0 \leq x \leq \pi \end{cases}$$

(a) **(10 points)** Use separation of variables $u(x,t) = X(x)T(t)$ to find a set of two second order differential equations satisfied by X and T. Please specify the boundary condition for X.

$$XT'' = X''T - 2T'$$

By separating X and T

$$\frac{X''}{X} = \frac{T''}{T} + \frac{2T'}{T} = -\lambda$$

Equation for X is

$$X'' + \lambda X = 0$$

The boundary conditions are

$$X(0) = 0 \text{ and } X(\pi) = 0$$

Equation for T is

$$T'' + 2T' + \lambda T = 0$$

Show your work

(b) (**10 points**) Find the displacement $u(x,t)$

The solution for

$$X'' + \lambda X = 0, X(0) = 0, X(\pi) = 0$$

are $\lambda = n^2, X_n(x) = \sin nx$, where $n = 1, 2, 3, \cdots$ The characteristic equation of

$$T'' + 2T' + n^2 T = 0 \text{ is } r^2 + 2r + n^2 = 0.$$

r is

$$r = -1 \pm \sqrt{1 - n^2}.$$

Since $n \geq 1$,

$$r = -1 \pm i\sqrt{n^2 - 1}.$$

Therefore

$$T(t) = K_1 e^{-t} \cos(t\sqrt{n^2 - 1}) + K_2 e^{-t} \sin(t\sqrt{n^2 - 1})$$

Since $T(0) = 0$

$$T(0) = K_1 e^0 \cos 0(\sqrt{n^2 - 1}) + K_2 e^0 \sin(0\sqrt{n^2 - 1}) = K_1 = 0.$$

Thus

$$T_n(t) = e^{-t} \sin(t\sqrt{n^2 - 1})$$

By multiplying X_n and T_n

$$u_n(x,t) = \sin nx e^{-t} \sin(t\sqrt{n^2 - 1})$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-t} \sin(t\sqrt{n^2 - 1})$$

$$u_t(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)(-e^{-t} \sin(t\sqrt{n^2 - 1}) + e^{-t}\sqrt{n^2 - 1} \cos t\sqrt{n^2 - 1})$$

$$u_t(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx)(-e^{0} \sin(0\sqrt{n^2 - 1}) + e^{0}\sqrt{n^2 - 1} \cos 0\sqrt{n^2 - 1})$$

$$= \sum_{n=1}^{\infty} c_n \sin(nx)\sqrt{n^2 - 1} = \sin 2x$$

By comparing both sides

$$c_1 = 0, c_2\sqrt{3} = 1, c_3 = 0, \cdots$$

Thus, $c_2 = \frac{1}{\sqrt{3}}$ and

$$u(x,t) = \frac{1}{\sqrt{3}} \sin 2x e^{-t} \sin(\sqrt{3}t)$$

Show your work

7. (a) (**2 points)** Draw graph of

$$f(x) = \begin{cases} 0, & -\pi < x \leq -\pi/2 \\ -2, & -\pi/2 < x \leq 0 \\ 2, & 0 < x \leq \pi/2 \\ 0, & \pi/2 < x \leq \pi \end{cases} \quad , \ f(x+2\pi) = f(x).$$

Part (b) next page →

Show your work

(b) (**9 points)** Find the Fourier series of $f(x)$.

Since $f(x)$ is an odd function, $a_0 = 0$ and $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx = \frac{1}{\pi}(\int_{-\pi/2}^{0} -2\sin nx dx + \int_{0}^{\pi/2} 2\sin nx dx)$$

$$= \frac{1}{\pi}(\frac{2}{n}[\cos nx]_{-\pi/2}^{0} - \frac{2}{n}[\cos nx]_{0}^{\pi/2}) = \frac{4}{n\pi}$$

Therefore

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n\pi}\sin nx$$

Show your work

8. (**10 points)** Either solve the boundary value problem, or show that it has no solution.

$$y'' + 5y = \cos(x),\ \ y'(0) = 0,\ \ y(\frac{\pi}{\sqrt{5}}) = 0.$$

Non-homogeneous equation

1. Homogeneous solution

$$y(x) = c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x$$

2. Particular solution
Let $y_p = A\cos x + B\sin x$, then $y_p'' = -A\cos x - B\sin x$. Substituting y_p and y_p'' gives

$$4A\cos x + 4B\sin x = \cos x$$

Thus, $A = \frac{1}{4}$, $B = 0$ and

$$y_p = \frac{1}{4}\cos x$$

3.General solution

$$y = c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x + \frac{1}{4}\cos x$$

$$y' = -c_1\sqrt{5}\sin\sqrt{5}x + c_2\sqrt{5}\cos\sqrt{5}x - \frac{1}{4}\sin x$$

$$y'(0) = c_2\sqrt{5}\cos(\sqrt{5}\cdot 0) = 0 \to c_2 = 0$$

$$y = c_1 \cos\sqrt{5}x + \frac{1}{4}\cos(x)$$

$$y(\frac{\pi}{\sqrt{5}}) = c_1 \cos\sqrt{5}\frac{\pi}{\sqrt{5}} + \frac{1}{4}\cos(\frac{\pi}{\sqrt{5}}) = 0 \to c_1 = \frac{1}{4}\cos(\frac{\pi}{\sqrt{5}})$$

Therefore

$$y = \frac{1}{4}\cos(\frac{\pi}{\sqrt{5}})\cos(\sqrt{5}x) + \frac{1}{4}\cos x$$

Show your work

9. (**11 points**) Solve the differential equation $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix}$. Eigenvalues

$$A - \lambda I = \begin{pmatrix} 5-\lambda & -3 \\ 3 & -1-\lambda \end{pmatrix}$$

$$det(A - \lambda I) = (1+\lambda)(\lambda - 5) + 9 = \lambda^2 - 4\lambda + 4 = 0 \rightarrow \lambda = 2$$

Eigenvectors

$$(A - 2I)\xi = \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \left(\begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \rightarrow \xi = \left(\begin{array}{c} 1 \\ 1 \end{array} \right)$$

Generalized eigenvector

$$(A - 2I)\eta = \xi \rightarrow \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) = \left(\begin{array}{c} 1 \\ 1 \end{array} \right)$$

Let $\eta_1 = k$, then $\eta_2 = -\frac{1}{3} + k$

$$\eta = \left(\begin{array}{c} 0 \\ -\frac{1}{3} \end{array} \right) + k \left(\begin{array}{c} 1 \\ 1 \end{array} \right)$$

Therefore, the solution is

$$x = C_1 e^{2t} \left(\begin{array}{c} 1 \\ 1 \end{array} \right) + C_2 (te^{2t} \left(\begin{array}{c} 1 \\ 1 \end{array} \right) + e^{2t} \left(\begin{array}{c} 0 \\ -\frac{1}{3} \end{array} \right))$$

Extra page for scratch work. I will not grade work on this page unless you write on another page“problem continued on page 14”.

Extra page for scratch work. I will not grade work on this page unless you write on another page "problem continued on page 15".