MATH 23 Exam 2 Review Solutions

Problem 1. *Use the method of reduction of order to find a second solution of the given differential equation*

$$x^2y'' - (x - 0.1875)y = 0, \quad x > 0, \quad y_1(x) = x^{1/4}e^{2\sqrt{x}}$$

Solution 1. *Set $y_2(x) = y_1(x)v(x)$, in which $y_1(x) = x^{1/4}e^{2x}$. It can be verified that y_1 is a solution of the differential equation, that is,*

$$x^2y'' - (x - 0.1875)y = 0.$$

Substitution of the given form of y_2 results in the differential equation

$$2x^{9/4}v'' + (4x^{7/4} + x^{5/4})v' = 0.$$

This equation is linear in the variable $w = v'$. An integrating factor is

$$\mu = e^{\int [2x^{-1/2}+1/(2x)]dx} = \sqrt{x}e^{4\sqrt{x}}.$$

Rewrite the equation as $[\sqrt{x}e^{4\sqrt{x}}v']' = 0$, from which it follows that $v'(x) = ce^{-4\sqrt{x}}/\sqrt{x}$. Integrating, $v(x) = c_1e^{-4\sqrt{x}} + c_2$ and as a result,

$$y_2(x) = c_1x^{1/4}e^{-2\sqrt{x}} + c_2x^{1/4}e^{2\sqrt{x}}.$$

Setting $c_1 = 1$ and $c_2 = 0$,we obtain

$$y_2(x) = x^{1/4}e^{-2\sqrt{x}}.$$

Problem 2. *Consider the initial value problem*

$$4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

(a) Solve the initial value problem and plot the solution.

(b) *Determine the coordinates* (t_M, y_M) *of the maximum point.*

(c) *Change the second initial condition to* $y'(0) = b > 0$ *and find the solution as a function of* b*.*

(d) *Find the coordinates* (t_M, y_M) *of the maximum point in terms of* b*. Describe the dependence of* t_M *and* y_M *on* b *as* b *increases.*

Solution 2. *(a) The characteristic equation is*

$$4r^2 + 4r + 1 = 0,$$

which is equivalent to

$$(2r + 1)^2 = 0.$$

The roots of the characteristic equation are

$$r = -1/2, -1/2.$$

So the solution is $y(t) = (c_1 + c_2 t)e^{-t/2}$*.*
Using the initial conditions, we get $c_1 = 1$ *and* $c_2 = 5/2$*. The solution becomes*

$$y(t) = (1 + \frac{5}{2})e^{-t/2}.$$

(b) *To determine the maximum point,*

$$y'(t) = 0 \implies e^{-t/2}\left(\frac{5}{2} - \frac{5}{4}t - \frac{1}{2}\right) = 0$$

which gives $t = \frac{8}{5}$*. Substituting this value of* t *into the solution, we get*

$$y_M = 5e^{-4/5}.$$

So, the maximum point $(t_M, y_M) = (\frac{8}{5}, 5e^{-4/5})$*.*

(c) *By changing the second initial condition from* $y'(0) = 2$ *to* $y'(0) = b$ *and keeping the first initial condition (i.e.,* $y(0) = 1$*) same, we get* $c_1 = 1$ *and* $c_2 = b + 1/2$ *and the solution becomes*

$$y(t) = (1 + (b + 1/2)t)e^{-t/2}.$$

(d) *To find the maximum point in terms of* b*, we differentiate the the solution obtained in part (c),*

$$e^{-t/2}(-\frac{1}{2} + (b + 1/2) - \frac{1}{2}(b + 1/2)t) = 0,$$

which gives

$$t_M = \frac{4b}{2b+1}.$$

Note that $t_M \to \infty$ *as* $b \to \infty$*.*
Substituting the value of t_M *into the solution gives*

$$y_M = (1 + 2b)exp(-\frac{2b}{1+2b}).$$

Again, note that $y_M \to \infty$ *as* $t \to \infty$*.*

Problem 3. *Solve the given initial value problem. Sketch the graph of the solution and describe its behaviour for increasing* t*.*

$$9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2$$

Solution 3. *The characteristic equation is*

$$9r^2 + 6r + 82 = 0.$$

We obtain the complex roots $r = -1/3 \pm 3i$*. The general solution is*

$$y(t) = c_1 e^{-t/3}\cos 3t + c_2 e^{-t/3}\sin 3t.$$

Based on the first initial condition, $c_1 = -1$*. Invoking the second initial condition, we conclude that* $1/3 + 3c_2 = 2$*, or* $c_2 = 5/9$*. Hence*

$$y(t) = -e^{-t/3}\cos 3t + (5/9)e^{-t/3}\sin 3t.$$

The solution oscillates with an exponentially decreasing amplitude.

Problem 4. *Find the general solution of the given differential equation*

$$u'' + \omega_0^2 u = \cos\omega t, \quad \omega^2 \neq \omega_0^2$$

Solution 4. *The characteristic equation for the homogeneous problem is*

$$r^2 + \omega_0^2 = 0,$$

with complex roots $r = \pm\omega_0 i$. *Hence*

$$y_c(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

Since $\omega \neq \omega_0$, *set*

$$Y = A\cos\omega t + B\sin\omega t.$$

Substitution into the ODE and comparing the coefficients results in the system of equations $(\omega_0^2 - \omega^2)A = 1$ *and* $(\omega_0^2 - \omega^2)B = 0$. *Hence*

$$Y = \frac{1}{\omega_0^2 - \omega^2}\cos\omega t.$$

The general solution is $y(t) = y_c(t) + Y$.

Problem 5. *Determine a suitable form of particular solution* $Y(t)$ *using the method of undetermined coefficients*

$$y'' + 3y' + 2y = e^t(t^2+1)\sin 2t + 3e^{-t}\cos t + 4e^t$$

Solution 5. *(a) The homogeneous solution is*

$$y_c(t) = c_1 e^{-t} + c_2 t e^{-2t}.$$

None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider

$$Y(t) = e^t(A_0+A_1t+A_2t^2)\cos 2t+e^t(B_0+B_1t+B_2t^2)\sin 2t++e^{-t}(C_1\cos t+C_2\sin t)+De^t.$$

(b) Substitution into the differential equation and comparing the coefficients results in

$$Y(t) = e^t(A_0+A_1t+A_2t^2)\cos 2t+e^t(B_0+B_1t+B_2t^2)\sin 2t+e^{-t}(-\frac{3}{2}\cos t+\frac{3}{2}\sin t)+2e^t/3,$$

in which $A_0 = -4105/35152, A_1 = 73/676, A_2 = -5/52, B_0 = -1233/35152, B_1 = 10/169, B_2 = 1/52.$

Problem 6. *Verify that the given functions y_1 and y_2 satisfy the corresponding homogenous equation; then find a particular solution of the given nonhomogeneous equation.*

$$x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0, \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$$

Solution 6. *Note that $g(x) = \ln x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1, y_2) = x^3$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)y_1(x)+u_2(x)y_2(x)$, in which*

$$u_1(x) = -\int \frac{x^2 \ln x(\ln x)}{W(x)} dx = -(\ln x)^3/3$$
$$u_2(x) = -\int \frac{x^2 \ln x}{W(x)} dx = -(\ln x)^2/2.$$

Therefore $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$.

Problem 7. *Verify that the given functions y_1 and y_2 satisfy the corresponding homogenous equation; then find a particular solution of the given nonhomogeneous equation.*

$$x^2y''+xy'+(x^2-0.25)y = g(x), \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x, \quad y_2(x) = x^{-1/2} \cos x$$

Solution 7. *First write the equation in standard form. The forcing function becomes $g(x)/x^2$. The functions $y_1(x) = x^{-1/2} \sin x$ and $y_2(x) = x^{-1/2} \cos x$ are a fundamental set of solutions. The Wronskian of the solutions is*

$$W(y_1, y_2) = -1/x.$$

Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, in which

$$u_1(x) = \int_{x_0}^{x} \frac{\cos \tau(g(\tau))}{\tau\sqrt{\tau}} d\tau,$$
$$u_2(x) = -\int_{x_0}^{x} \frac{\sin \tau(g(\tau))}{\tau\sqrt{\tau}} d\tau.$$

Therefore

$$\begin{aligned} Y(x) &= \frac{\sin x}{\sqrt{x}} \int_{x_0}^{x} \frac{\cos\tau(g(\tau))}{\tau\sqrt{\tau}} d\tau - \frac{\cos x}{\sqrt{x}} \int_{x_0}^{x} \frac{\sin\tau(g(\tau))}{\tau\sqrt{\tau}} d\tau \\ &= \frac{1}{\sqrt{x}} \int_{x_0}^{x} \frac{\sin(x-\tau)(g(\tau))}{\tau\sqrt{\tau}} d\tau. \end{aligned}$$

Problem 8. *A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.*

Solution 8. *The spring constant is* $k = \frac{3}{(1/4)} = 12$ *lb/ft. Mass* $m = 3/32$ *lb-s*^{2} */ft. Since there is no damping, the equation of motion is*

$$3u''/32 + 12u = 0,$$

that is, $u'' + 128u = 0$*. The initial conditions are* $u(0) = -1/12$ *ft ,* $u'(0) = 2$ *ft/s. The general solution is*

$$u(t) = A\cos(8\sqrt{2}t) + B\sin(8\sqrt{2}t).$$

Invoking the initial conditions, we have

$$u(t) = -\frac{1}{12}\cos(8\sqrt{2}t) + \frac{1}{4\sqrt{2}}\sin(8\sqrt{2}t).$$

$R = \sqrt{11/288}$*ft,* $\delta = \pi - \arctan(3/\sqrt{2})$*rad,* $\omega_0 = 8\sqrt{2}$ *rad/s,* $T = \pi/(4\sqrt{2})$*s.*

Problem 9. *A 1/4-kg mass is attached to a spring with a stiffness 4 N/m. The damping constant b for the system is 1 N-sec/m. If the mass is displaced 1/2 m to the left and an initial velocity of 1 m/sec to the left, find the equation of motion. What is the maximum displacement that the mass will attain?*

Solution 9. *The general equation is*

$$my'' + by' + ky = 0 \tag{1}$$

Substituting $m = 1/4, b = 1, k = 4$ *into Eq (1), we get*

$$(1/4)y'' + y' + 4y = 0$$

with initial conditions

$$y(0) = -1/2, \quad y'(0) = -1.$$

The negative signs for the initial conditions reflect the facts that the initial displacement and push are to the left. The solution to (1) is given by

$$y(t) = -\frac{1}{2}e^{-2t}\cos(2\sqrt{3}t) - \frac{1}{\sqrt{3}}e^{-2t}\sin(2\sqrt{3}t)$$

or, it can be expressed as

$$y(t) = \sqrt{\frac{7}{12}}e^{-2t}\sin(2\sqrt{3}t + \phi),$$

where $\phi = \sqrt{3}/2$ *and* ϕ *lies in Quadrant III because* $c_1 = -1/2$ *and* $c_2 = -1/\sqrt{3}$ *are both negative.*
To determine the maximum displacement from equilibrium, we must determine the maximum value of $|y(t)|$.

$$y'(t) = e^{-2t}\{\frac{5}{\sqrt{3}}\sin(2\sqrt{3}t) - \cos(2\sqrt{3}t)\} = 0,$$

$$\frac{5}{\sqrt{3}}\sin(2\sqrt{3}t) = \cos(2\sqrt{3}t),$$

$$\tan(2\sqrt{3}t) = \frac{\sqrt{3}}{5}.$$

Thus, the first positive root is

$$t = \frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}}{5} \approx 0.096.$$

Substituting this value for t back into the solution $y(t)$ *gives* $y(0.096) \approx -0.55$. *Hence the maximum displacement, which occurs to the left of equilibrium, is approximately 0.55m.*

Problem 10. *Verify that the given vector satisfies the given differential equation*

$$x' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} x, \quad x = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} e^{-t} + 2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-2t}$$

Solution 10. *It is easy to see that*

$$x' = \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} e^{2t} = \begin{bmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t}\ 4e^{-t} - 4e^{-2t} \end{bmatrix}$$

On the other hand,

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} x &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} e^{2t} \\ &= \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} e^{2t} \end{aligned}$$

Problem 11. *Verify that the given matrix satisfies the given differential equation*

$$\psi' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \psi, \quad \psi(t) = \begin{bmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{bmatrix}$$

Solution 11. *Differentiation, elementwise, results in*

$$\psi' = \begin{bmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{bmatrix}$$

On the other hand,

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \psi &= \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{bmatrix} \\ &= \begin{bmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{bmatrix}. \end{aligned}$$

Problem 12. *Verify that the given functions are solutions of the differential equation, and determine the Wronskian*

$$xy''' - y'' = 0; \quad 1, \quad x, \quad x^3$$

Solution 12. *Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have* $W(1, x, x^3) = 6x$.

Problem 13. *Determine intervals in which solutions are sure to exist*

$$(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$$

Solution 13. *Writing the equation in standard form, the coefficients are rational functions with a singularity at* $x_0 = 1$. *Furthermore,* $p_4(x) = \tan x/(x-1)$ *is undefined, and hence not continuous, at* $x_k = \pm(2k+1)\pi/2, k = 0, 1, 2, \ldots$ *Hence solutions are defined on any interval that does not contain* x_0 *or* x_k.

Problem 14. *Find the general solution of the given differential equation*

$$y^{(6)} - y'' = 0$$

Solution 14. *The characteristic equation can be written as*

$$r^2(r^4 - 1) = 0.$$

The roots are given by

$$r = 0, 0, \pm 1, \pm i.$$

The general solution is

$$y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^t + c_5 \cos t + c_6 \sin t.$$

Problem 15. *Find the general solution of the given differential equation*

$$y^{(8)} + 8y^{(4)} + 16y = 0$$

Solution 15. *The characteristic equation can be written as*

$$(r^4 + 4)^2 = 0.$$

The roots of the equation $r^4 + 4 = 0$ *are*

$$r = 1 \pm i, -1 \pm i.$$

Each of these roots has multiplicity two. The general solution is

$$y = e^t[c_1 \cos t + c_2 \sin t] + te^t[c_3 \cos t + c_4 \sin t] + e^{-t}[c_5 \cos t + c_6 \sin t] + te^{-t}[c_7 \cos t + c_8 \sin t].$$

Problem 16. *Determine a suitable form for the particular solution ($Y(t)$) if the method of undetermined coefficients is to be used. Do not evaluate the constants.*

$$y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$$

Solution 16. *The characteristic equation can be written as*

$$r^2(r^2 + 2r + 2) = 0,$$

with roots $r = 0$, with multiplicity two, and $r = -1 \pm i$. This means that the homogeneous solution is

$$y_c = c_1 + c_2 t + c_3 e^{-t}\cos t + c_4 e^{-t}\sin t.$$

The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set

$$Y_1(t) = Ae^t + (Bt + C)e^{-t}.$$

Now $g_2(t) = e^{-t}\sin t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set

$$Y_2(t) = t(De^{-t}\cos t + Ee^{-t}\sin t).$$

It follows that the particular solution has the form

$$Y(t) = Ae^t + (Bt + C)e^{-t} + t(De^{-t}\cos t + Ee^{-t}\sin t).$$

Problem 17. *Find the solution of the initial value problem*

$$y''' - 3y'' + 2y' = t + e^t; \quad y(0) = 1, \quad y'(0) = -1/4, \quad y''(0) = -3/2$$

Solution 17. *The characteristic equation can be written as*

$$r(r^2 - 3r + 2) = 0.$$

Hence the homogeneous solution is

$$y_c = c_1 + c_2 e^t + c_3 e^{2t}.$$

Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set

$$Y_1(t) = Ate^t.$$

Substitution into the ODE results in $A = -1$. *Now let*

$$Y_2(t) = Bt^2 + Ct.$$

Substitution into the ODE results in $B = 1/4$ *and* $C = 3/4$. *Therefore the general solution is*

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + (t^2 + 3t)/4.$$

Invoking the initial conditions, we find that $c_1 = 1, c_2 = c_3 = 0$. *The solution of the initial value problem is*

$$y(t) = 1 - te^t + (t^2 + 3t)/4.$$

Problem 18. *Given that* x, x^2, *and* $1/x$ *are solutions of the homogeneous equation corresponding to*

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4, \quad x > 0,$$

determine a particular solution.

Solution 18. *First write the equation as*

$$y''' + x^{-1} y'' - 2x^{-2} y' + 2x^{-3} y = 2x.$$

The Wronskian is evaluated as

$$W(x, x^2, 1/x) = 6/x.$$

Now compute the three determinants

$$W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3,$$

$$W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x,$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

Hence $u_1(x) = -x^3/3, u_2(x) = x^2/3, u_3(x) = x^5/15$. *Therefore the particular solution can be expressed as*

$$Y(x) = x[-x^3/3] + x^2[x^2/3] + \frac{1}{x}[x^5/15] = x^4/15.$$

Problem 19. *Find the solution of the given initial value problem*

$$y''' + y' = \sec t; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -2$$

Solution 19. *The characteristic equation is*

$$r(r^2 - 1) = 0$$

and the roots are $r = 0, \pm i$. The complementary solution is given by

$$y_c(t) = c_1 + c_2 \cos t + c_3 \sin t.$$

The Wronskian $W(1, \cos t, \sin t) = 1$ and

$$W_1 = 1, W_2 = -\cos t, W_3 = -\sin t$$

This gives

$$\begin{aligned} u_1(t) &= \int \frac{W_1 \sec t}{W} dt = \ln|\sec t + \tan t| \\ u_2(t) &= \int \frac{W_2 \sec t}{W} dt = -t \\ u_3(t) &= \int \frac{W_3 \sec t}{W} dt = \ln|\sec t| \end{aligned}$$

The particular solution is given by

$$Y(t) = u_1(t) + u_2(t)\cos t + u_3(t)\sin t,$$

and using u_1, u_2 and u_3, the solution becomes

$$Y(t) = \ln|\sec t + \tan t| - t\cos t + \ln|\sec t|\sin t.$$

Problem 20. *Find all eigenvalues and eigenvectors of the given matrix*

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Solution 20. *The characteristic equation is given by*

$$(1 - \lambda)^2 - 1 = 0$$

which gives $\lambda = 0, 2$. *For* $\lambda_1 = 0$

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \sim \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$$

which gives $-ix_1 + x_2 = 0$. *So, the eigenvector is*

$$x^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

For $\lambda_1 = 2$

$$\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \sim \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$$

which gives $-ix_1 - x_2 = 0$. *So, the eigenvector is*

$$x^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Problem 21. *Find all eigenvalues and eigenvectors of the given matrix*

$$\begin{bmatrix} 11/9 & -2/9 & 8/9 \\ -2/9 & 2/9 & 10/9 \\ 8/9 & 10/9 & 5/9 \end{bmatrix}$$

Solution 21. *For computational purposes, note that if* λ *is an eigenvalue of* B, *then* $c\lambda$ *is an eigenvalue of the matrix* $A = cB$. *Eigenvectors are unaffected, since they are only determined up to a scalar multiple. So with the associated characteristic equation is*

$$\mu^3 - 18\mu^2 - 81\mu + 1458 = 0,$$

with roots $\mu_1 = -9, \mu^2 = 9$ *and* $\mu_3 = 18$. *Hence the eigenvalues of the given matrix,* A, *are* $\lambda_1 = -1, \lambda_2 = 1$ *and* $\lambda_3 = 2$. *Setting* $\lambda = \lambda_1 = -1$, *(which corresponds to using* $\mu_1 = -9$ *in the modified problem) the reduced system of equations is*

$$2x_1 + x_3 = 0,$$
$$x_2 + x_3 = 0.$$

A corresponding solution vector is given by $x^{(1)} = (1, 2, -2)^T$. *Setting* $\lambda = \lambda_2 = 1$, *the reduced system of equations is*

$$\begin{aligned} x_1 + 2x_3 &= 0, \\ x_2 - 2x_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $x^{(2)} = (2, -2, -1)^T$. *Finally, setting* $\lambda = \lambda_2 = 1$, *the reduced system of equations is*

$$\begin{aligned} x_1 - x_3 &= 0, \\ 2x_2 - x_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $x^{(3)} = (2, 1, 2)^T$.