MATH 23 EXAM 1 REVIEW PROBLEMS

Problem 1. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Solution. Let $V(t)$ and $S(t)$ be the volume and surface area, respectively, of the raindrop at time t. Then

$$V(t) = \frac{4}{3}\pi r(t)^3 \qquad S(t) = 4\pi r(t)^2$$

where $r(t)$ is the radius of the drop at time t. Since the rate of evaporation is proportional to the surface area, then

$$\frac{dV}{dt} = \alpha S$$

for some constant α. We now express S in terms of V. Observe that

$$S^{3/2} = (4\pi r^2)^{3/2} = 8\pi^{3/2} r^3$$

so

$$\frac{S^{3/2}}{V} = \frac{8\pi^{3/2} r^3}{\frac{4}{3}\pi r^3} = 6\pi^{1/2}\,.$$

Then

$$S^{3/2} = 6\pi^{1/2} V \implies S = 6^{2/3}\pi^{1/3} V^{2/3} = (36\pi)^{1/3} V^{2/3}$$

so

$$\frac{dV}{dt} = \alpha S = \alpha (36\pi)^{1/3} V^{2/3}\,.$$

(You are not required to solve this equation in this problem, but you can using separation of variables, which gives

$$V = (\alpha(36\pi)^{1/3} t + {V_0}^{1/3})^3$$

where V_0 is the initial volume.)

Problem 2. For large, rapidly falling objects, the drag force is approximately proportional to the square of the velocity.

(a) Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.
(b) Determine the limiting velocity after a long time.
(c) If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.
(d) Using the data in part (c), draw a direction field for the differential equation.

Solution. (a) We orient our axes so that down is the positive direction. Then the two forces acting on the object are the force due to gravity, which is mg (where $g \approx 9.81\text{m/s}^2$ is the acceleration due to gravity) and the force due to drag, which is $-kv$ for some positive constant k. By Newton's second law of motion, then

$$ma = mv' = mg - kv^2$$

where a is the object's acceleration. Thus our equation can be written

$$v' = g - (k/m)v^2 .$$

(b) We seek to find the equilibrium solution, i.e., a function $v(t)$ that makes $v'(t)$ identically zero. We find

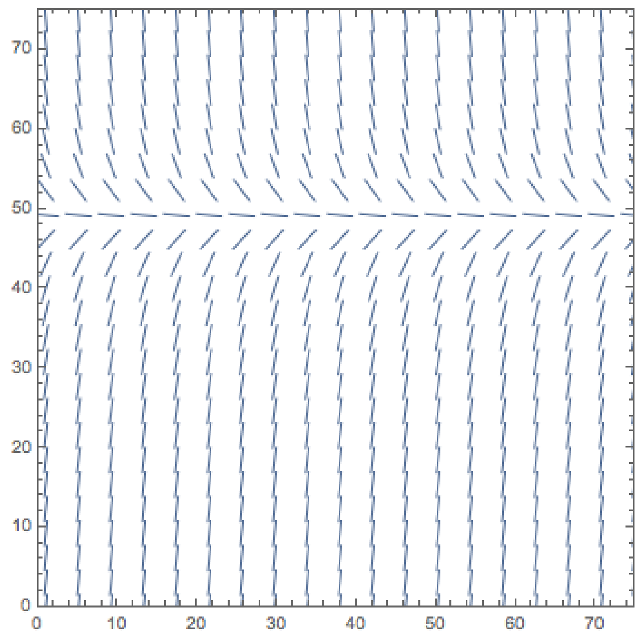
$$0 = v' = g - (k/m)v^2 \implies (k/m)v^2 = g \implies v^2 = \frac{mg}{k} \implies v = \sqrt{\frac{mg}{k}} .$$

Thus after a long time, we expect the object to have velocity $\sqrt{mg/k}$.

(c)

$$49 = \sqrt{10g/k} \implies 49^2 k = 10g \implies k = \frac{10g}{2401} \approx \frac{98.1}{2401} \approx 0.0409 .$$

(d)



Problem 3. According to Newton's law of cooling, the temperature $u(t)$ of an object satisfies the differential equation

$$\frac{du}{dt} = -k(u - T)$$

where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.

(a) Find the temperature of the object at any time t.

(b) Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

Solution. (a) Separating variables, we have

$$\frac{du}{dt} = -k(u - T) \implies \ln|u - T| = \int \frac{du}{u - T} = -\int k\,dt = -kt + C_0 .$$

Exponentiating, then

$$u - T = Ce^{kt} \implies u = T + Ce^{-kt}.$$

The initial condition implies

$$u_0 = u(0) = T + C \implies C = u_0 - T$$

which gives a final answer of

$$u = T + (u_0 - T)e^{-kt}$$

(b) By the definition of τ, we have

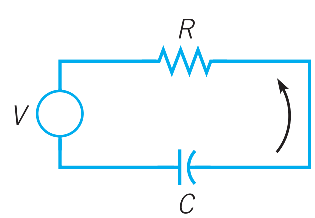
$$\frac{1}{2}(u_0 - T) = u(\tau) - T = (u_0 - T)e^{-k\tau} \implies \frac{1}{2} = e^{-k\tau} \implies e^{k\tau} = 2 \implies k\tau = \ln(2).$$

Problem 4. Consider an electric circuit containing a capacitor, resistor, and battery; see the figure below. The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

(a) If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus r.
(b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.



Solution. (a) Separating variables, we have

$$\frac{dQ}{dt} = \frac{1}{R}\left(V - \frac{Q}{C}\right) = \frac{1}{RC}(CV - Q)$$

$$\implies -\ln|CV - Q| = \int \frac{dQ}{CV - Q} = \int \frac{1}{C}\,dt = \frac{1}{RC}t + C_0.$$

Exponentiating both sides yields

$$CV - Q = C_1 e^{-\frac{1}{RC}t} \implies Q = CV - C_1 e^{-\frac{1}{RC}t}.$$

The initial condition implies

$$0 = Q(0) = CV - C_1 \implies C_1 = CV$$

so we get a final answer of

$$Q(t) = CV - CVe^{-\frac{1}{RC}t} = CV(1 - e^{-\frac{1}{RC}t})$$

(b) Setting $dQ/dt = 0$, we find

$$0 = R \cdot 0 = V - \frac{Q_L}{C} \implies Q_L = CV$$

where Q_L is the limiting value.

Problem 5. Find the solution to the initial value problem

$$y' + 2y = te^{-2t}, \qquad y(1) = 0\,.$$

Solution. The integrating factor is $\mu(t) = e^{2t}$. Multiplying both sides of the equation by μ, we have

$$\frac{d}{dt}[e^{2t}y] = t\,.$$

Integrating both sides of the equation results in the general solution $y(t) = (1/2)t^2e^{-2t} + ce^{-2t}$. The initial condition implies that $(1/2)e^{-2} + ce^{-2} = 0$, hence $c = -1/2$, so the solution of the IVP is

$$y(t) = \frac{1}{2}(t^2 - 1)e^{-2t}\,.$$

Problem 6. Consider the initial value problem

$$y' + \frac{1}{2}y = 2\cos(t), \qquad y(0) = -1\,.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

Solution. The integrating factor is

$$\mu(t) = e^{\int (1/2)dt} = e^{t/2}\,.$$

Therefore the general solution is

$$y(t) = \frac{1}{5}(4\cos(t) + 8\sin(t)) + ce^{-t/2}\,.$$

The initial condition gives a specific solution

$$y(t) = \frac{1}{5}(4\cos(t) + 8\sin(t) - 9e^{-t/2})\,.$$

Differentiating, we find

$$y'(t) = \frac{1}{5}(-4\sin(t) + 8\cos(t) + (9/2)e^{-t/2}$$
$$y''(t) = \frac{1}{5}(-4\cos(t) - 8\sin(t) - (9/4)e^{-t/2}\,.$$

Setting $y'(t) = 0$, the first critical value is $t_1 \approx 1.3643$. Since $y''(t_1) < 0$, this is a local maximum with approximate coordinates $(1.3643, 0.82008)$.

Problem 7. Consider the initial value problem

$$y' - \frac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0\,.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

Solution. The integrating factor is $\mu(t) = e^{-3t/2}$. Multiplying through, we have

$$\frac{d}{dt}[e^{-3t/2}y] = 3te^{-3t/2} + 2e^{-t/2}.$$

Integrating, the general solution is

$$y(t) = -2t - 4/3 - 4e^t + ce^{3t/2}.$$

The initial condition implies

$$y(t) = -2t - 4/3 - 4e^t + (y_0 + 16/3)e^{3t/2}.$$

Now, as $t \to \infty$ the $(y_0 + 16/3)e^{3t/2}$ term will dominate, hence its sign will determine whether the solution grows positively or negatively. Thus the critical value of the initial condition is $y_0 = -16/3$. The corresponding solution

$$y(t) = -2t - 4/3 - 4e^t$$

decreases without bound.

Problem 8. Consider the initial value problem

$$y' = \frac{x(x^2+1)}{4y^3}, \qquad y(0) = -\frac{1}{\sqrt{2}}.$$

(a) Solve the IVP.
(b) Determine the interval on which the solution is valid.

Solution. (a) Rewriting the differential equation as $4y^3 dy = x(x^2+1)dx$, integration yields

$$y^4 = \frac{1}{4}(x^2+1)^2 + c.$$

The initial condition implies $c = 0$, which gives an implicit solution $(x^2+1)^2 - 4y^4 = 0$. The explicit form is then

$$y(x) = -\sqrt{\frac{x^2+1}{2}}$$

where we have chosen the sign so that $y(0) = -1/\sqrt{2}$.

(b) Since $x^2 + 1 \geq 1$ for all x, then the solution is valid for all x, so $(-\infty, \infty)$ is the interval of validity.

Problem 9. Consider the initial value problem

$$y' = \frac{ty(4-y)}{1+t}, \qquad y(0) = y_0 > 0.$$

(a) Determine how the solution behaves as $t \to \infty$.
(b) If $y_0 = 2$, find the approximate time T at which the solution first reaches the value 3.99.

Solution. (a) Separating variables, we have

$$\frac{dy}{y(4-y)} = \frac{t}{1+t}\,dt\,.$$

The partial fraction decomposition of the LHS is

$$\frac{1}{y(4-y)} = \frac{1/4}{y} - \frac{1/4}{y-4}$$

so we have

$$\ln\left|\frac{y}{y-4}\right| = \ln|y| - \ln|y-4| = \int\left(\frac{1}{y} - \frac{1}{y-4}\right)dy = 4\int\frac{t}{1+t}\,dt = 4t - 4\ln|1+t| + c\,.$$

Exponentiating yields

$$\frac{y}{y-4} = c_1\frac{e^{4t}}{(1+t)^4}\,.$$

We solve for y:

$$\begin{aligned}
y = (y-4)c_1\frac{e^{4t}}{(1+t)^4} &= yc_1\frac{e^{4t}}{(1+t)^4} - 4c_1\frac{e^{4t}}{(1+t)^4} \\
\implies y\left(1 - c_1\frac{e^{4t}}{(1+t)^4}\right) &= -4c_1\frac{e^{4t}}{(1+t)^4} \\
\implies y = -4c_1\frac{\frac{e^{4t}}{(1+t)^4}}{1 - c_1\frac{e^{4t}}{(1+t)^4}} &= \frac{-4c_1e^{4t}}{(1+t)^4 - c_1e^{4t}} = \frac{4c_1e^{4t}}{c_1e^{4t} - (1+t)^4}\,.
\end{aligned}$$

Thus as $t \to \infty$,

$$y(t) = \frac{4c_1e^{4t}}{c_1e^{4t} - (1+t)^4} = \frac{4}{1 - \frac{(1+t)^4}{c_1e^{4t}}} \to \frac{4}{1-0} = 4\,.$$

(b) The initial condition $y(0) = 2$ implies

$$2 = y(0) = \frac{4c_1}{c_1 - 1} \implies 2c_1 - 2 = 4c_1 \implies -2 = 2c_1 \implies c_1 = -1$$

so

$$y(t) = \frac{-4e^{4t}}{-e^{4t} - (1+t)^4} = \frac{4e^{4t}}{e^{4t} + (1+t)^4}\,.$$

Using a computer, we find the numerical answer to

$$3.99 = y(t) = \frac{4e^{4t}}{e^{4t} + (1+t)^4}$$

is $t \approx 2.84367$.

Problem 10. Using a theorem proved in class, determine an interval on which the solution of the following initial value problem is guaranteed to exist. Be sure to state how you are using the theorem.

$$(\ln(t))y' + y = \cot(t), \qquad y(2) = 3$$

Solution. Rewriting the equation in standard form, we have

$$y' + \frac{1}{\ln(t)} y = \frac{\cos(t)}{\sin(t)}\,.$$

Since $\ln(1) = 0$ and $\sin(t) = 0$ for integer multiples of π, then the coefficient functions are discontinuous there. The interval containing the initial value $t_0 = 2$ is $(1, \pi)$, which is the desired interval.

Problem 11. State where in the t, y-plane the hypotheses of Theorem 2.4.2 are satisfied for the following ODE.

$$\frac{dy}{dt} = \frac{(\cot(t))y}{1+y}\,.$$

Solution. The function $f(t, y)$ is discontinuous along the lines $t = \pm k\pi$ for $k = 0, 1, 2, \ldots$ and $y = -1$. The partial derivative

$$f_y = \frac{\cot(t)}{(1+y)^2}$$

has the same discontinuities. Thus the hypotheses are verified for all points in the t, y-plane not mentioned above.

Problem 12. Heat transfer from a body to its surrounds by radiation, based on the Stefan-Boltzmann law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4)\,,$$

where $u(t)$ is the absolute temperature of the body at time t, T is the absolute temperature of the surroundings, and α is a constant. However if u is much larger than T, the solutions to the above equation are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4\,. \tag{1}$$

Suppose that a body with initial temperature 2000 K is surrounded by a medium with temperature 300 K and that $\alpha = 2 \cdot 10^{-12} \frac{1}{\mathrm{K}^3\mathrm{s}}$.

(a) Determine the temperature of the body at any time t by solving (1).
(b) Describe the solution's behavior as $t \to \infty$. Does this behavior make sense?

Solution. The equation is separable and has implicit solution

$$u^3 = \frac{u_0^3}{3\alpha u_0^3 t + 1}\,.$$

With the given values, it follows that

$$u(t) = \frac{2000}{\sqrt[3]{\frac{6t}{125} + 1}}\,.$$

(a) As $t \to \infty$, we see that $u(t) \to 0$. This doesn't make sense: we expect the body's temperature to approach the ambient temperature of 300 K. This is because we didn't use the true Stefan-Boltzmann law

$$\frac{du}{dt} = -\alpha(u^4 - T^4)\,.$$

Problem 13. In this problem, we will attempt to find the curve along which a particle will slide without friction in the minimum time from a given point P to another point Q that is lower than P.

It is convenient to take the upper point P as the origin, and to orient the axes as shown below. The lower point Q has coordinates (x_0, y_0). One can show that the desired curve is given by a function $y(t)$ satisfying the differential equation

$$(1 + (y')^2)y = k^2$$

where k^2 is some positive constant.

(a) Solve the above equation for y'. Why is it necessary to choose the positive square root?

(b) Introduce the new variable t given by the equation

$$y = k^2 \sin^2(t)\,.$$

Show that the equation found in part (a) then takes the form

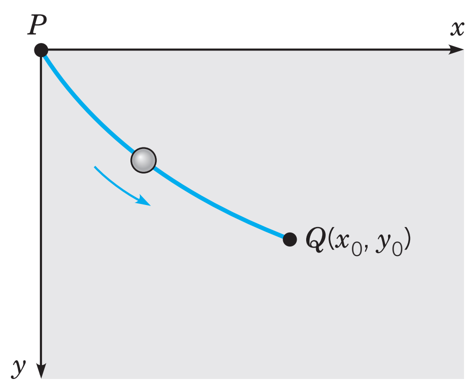
$$2k^2 \sin^2(t)\, dt = dx\,. \tag{2}$$

(c) Letting $\theta = 2t$, show that the solution of (2) for which $x = 0$ when $y = 0$ is given by

$$x(\theta) = \frac{k^2(\theta - \sin\theta)}{2}, \qquad y(\theta) = \frac{k^2(1 - \cos\theta)}{2}\,.$$

(The graph of the equations $(x(\theta), y(\theta))$ is called a *cycloid*.)

(d) If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of to the problem described at the beginning of the exercise. Find k if $x_0 = 1$ and $y_0 = 2$.



Solution. (a) Solving the equation for y', we find

$$y' = \sqrt{\frac{k^2}{y} - 1} = \sqrt{\frac{k^2 - y}{y}}\,.$$

We choose the positive root because of how we have oriented our axes: y is a nonnegative, increasing function of x.

(b) Letting $y = k^2 \sin^2(t)$, then $dy = 2k^2 \sin(t)\cos(t)\,dt$. Note that

$$\frac{k^2 - y}{y} = \frac{k^2 - k^2\sin^2(t)}{k^2\sin^2(t)} = \frac{\cos^2(t)}{\sin^2(t)}\,.$$

Substituting these two expressions into the result of part (a), we have

$$\frac{\cos(t)}{\sin(t)} = \sqrt{\frac{k^2-y}{y}} = \frac{dy}{dx} = \frac{dy = 2k^2\sin(t)\cos(t)\,dt}{dx}$$

hence

$$2k^2\sin^2(t)\,dt = dx\,.$$

(c) Setting $\theta = 2t$, then $d\theta = 2\,dt$, so we obtain $k^2\sin^2(\theta/2)\,d\theta = dx$. Integrating both sides and noting that $t = \theta = 0$ corresponds to the origin, we find

$$x(\theta) = \frac{k^2(\theta - \sin(\theta))}{2}\,.$$

To find y, we recall the expression from part (b):

$$y = k^2\sin^2(t) = k^2\frac{1-\cos(2t)}{2} = \frac{k^2(1-\cos(\theta))}{2}\,.$$

by the half-angle formula.

(d) Observe that

$$\frac{y}{x} = \frac{1-\cos(\theta)}{\theta - \sin(\theta)}\,.$$

Setting $x = 1, y = 2$, we find the approximate solution of

$$2 = \frac{1-\cos(\theta)}{\theta-\sin(\theta)}$$

is $\theta \approx 1.401$. Substituting this into the equation for $x(\theta)$ and solving for k yields an approximate solution of $k \approx 2.193$.

Problem 14. Consider the autonomous ODE below. Let $f(y) = dy/dt$.

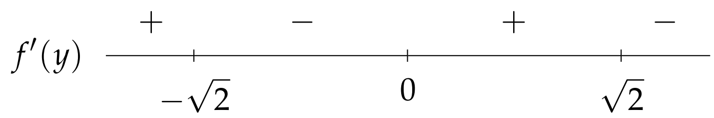
$$\frac{dy}{dt} = y^2(4-y^2) \qquad -\infty < y_0 < \infty$$

(a) Sketch the graph of $f(y)$ vs. y by hand (i.e., without using a graphing utility, such as a graphing calculator).
(b) Determine the equilibria, and classify each one as stable, unstable, or semistable.
(c) Draw the phase line, and sketch several graphs of solutions in the t, y-plane.

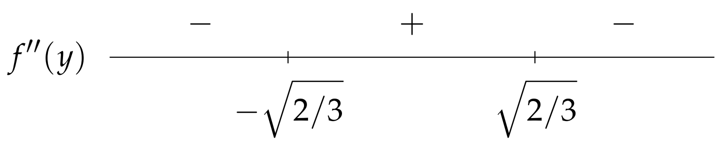
Solution. (a) Since $f(y) = y^2(2-y)(2+y) = 4y^2 - y^4$, then f has zeroes at $0, \pm 2$. We compute

$$f'(y) = 8y - 4y^3 = 4y(2-y^2)$$
$$f''(y) = 8 - 12y^2\,.$$

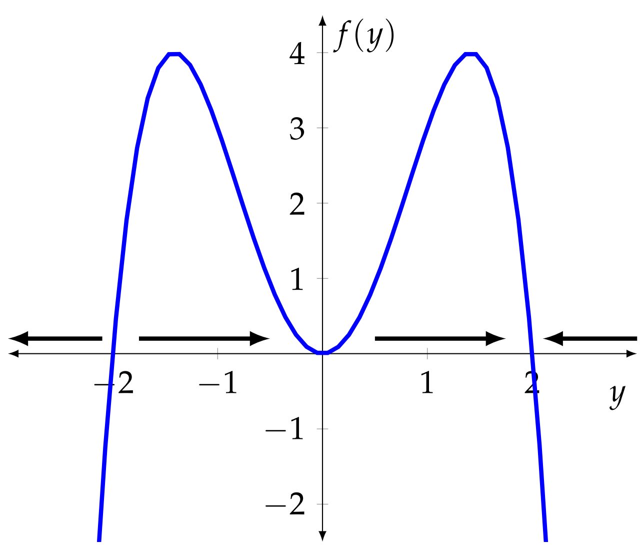
Choosing test points, we have the following sign chart for $f'(y)$



and the following sign chart for $f''(y)$.

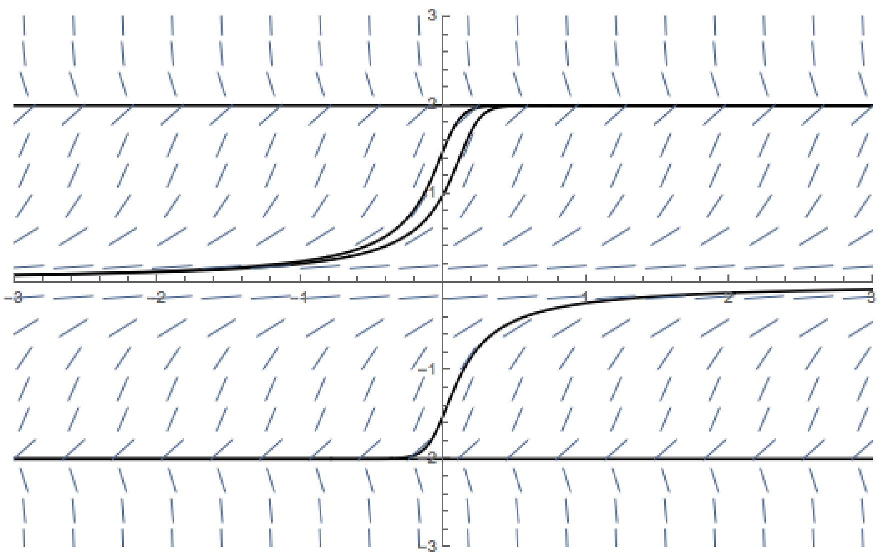


With this information about where the f is increasing and decreasing and its concavity, we can make a reasonably accurate plot of f.



(b) As we remarked above, $f(y) = dy/dt$ has zeroes at $y = -2, 0, 2$ which are the equilibria. From the phase line sketch, we see that $y = -2$ is unstable, $y = 0$ is semistable and $y = 2$ is stable.

(c)

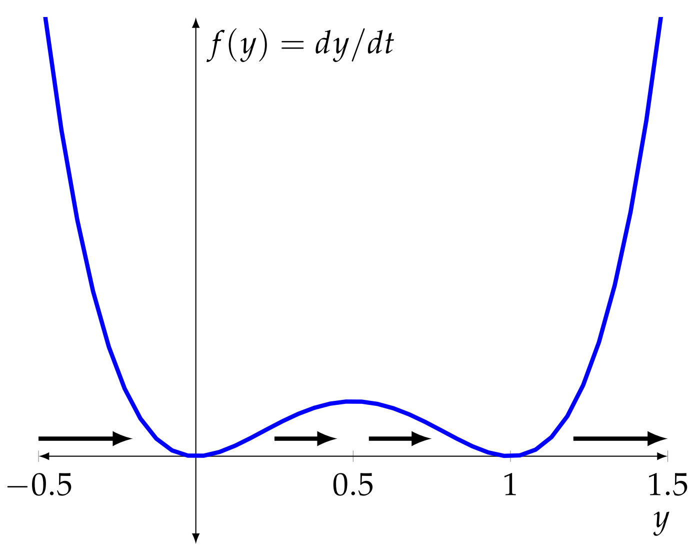


Problem 15. Consider the autonomous ODE below. Let $f(y) = dy/dt$.

$$\frac{dy}{dt} = y^2(1-y)^2 \qquad -\infty < y_0 < \infty$$

(a) Sketch the graph of $f(y)$ vs. y by hand (i.e., without using a graphing utility, such as a graphing calculator).
(b) Determine the equilibria, and classify each one as stable, unstable, or semistable.
(c) Draw the phase line, and sketch several graphs of solutions in the t, y-plane.

Solution. (a) Using the same steps as outlined in the previous problem, we construct the following plot.



(b) The equilibria $y = 0$ and $y = 1$ are both semistable, as can be seen from the phase line.

Problem 16. Determine whether the following equation is exact. If it is exact, find the solution.

$$(2x+3)\,dx + (2y-2)\,dy = 0\,.$$

Solution. Since $M_y = 0 = N_x$, then the equation is exact. Then there is some function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$. Then

$$\psi = \int (2x+3)\,dx = x^2 + 3x + C(y)$$

so $\psi_y = C'(y)$. Comparing this with N, we find $C'(y) = N = 2y - 2$, so $C(y) = y^2 - 2y + C_0$. Taking $C_0 = 0$, then $\psi = x^2 + 3x + y^2 - 2y$, so the equation can be written as

$$\frac{d}{dx}\psi = \frac{d}{dx}(x^2 + 3x + y^2 - 2y) = 0$$

which has solution $x^2 + 3x + y^2 - 2y = c$.

Problem 17. Find an integrating factor and solve the following equation.

$$y + (2x - ye^y)y' = 0\,.$$

Solution. $M_y = 1$ and $N_x = 2$, so we take as integrating factor

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\,dy\right) = \exp\left(\int \frac{dy}{y}\right) = e^{\ln(y)} = y\,.$$

Multiplying through by μ, we obtain

$$y^2\,dx + (2xy - y^2e^y)\,dy = 0\,.$$

Redefining M and N as the coefficient functions of this new equation, we have $\psi_x = M$ and $\psi_y = N$. Then

$$\psi = \int y^2\,dx = xy^2 + C(y) \implies \psi_y = 2xy + C'(y)$$

Since $\psi_y = N = 2xy - y^2e^y$, we find $C'(y) = -y^2e^y$. Using integration by parts, we find $C(y) = -e^y(y^2 - 2y + 2)$, so

$$\psi = xy^2 - e^y(y^2 - 2y + 2)$$

and our equation has solution

$$xy^2 - e^y(y^2 - 2y + 2) = c\,.$$

Problem 18. Solve the following equation.

$$(3x^2y + 2xy + y^3)\,dx + (x^2 + y^2)\,dy = 0\,.$$

Solution. Computing the relevant partials, we find

$$M_y = 3x^2 + 2x + 3y^2 \qquad N_x = 2x\,.$$

Then $\frac{M_y - N_x}{N} = 3$ is a function of x alone (a constant, actually), so we can compute an integrating factor as follows.

$$\mu(x) = e^{\int 3\,dx} = e^{3x}.$$

Multiplying through by $\mu(x)$, we have

$$\overbrace{(3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x})}^{M}\,dx + \overbrace{(x^2e^{3x} + y^2e^{3x})}^{N}\,dy = 0$$

where we have redefined M and N. This equation is exact, so $\psi_y = N$ for some function $\psi(x, y)$. We determine ψ by integrating N with respect to y:

$$\psi = \int N\,dy = \int (x^2e^{3x} + y^2e^{3x})\,dy = x^2ye^{3x} + \frac{y^3}{3}e^{3x} + C(x)$$

for some function $C(x)$. We also know $\psi_x = M$, and taking the partial with respect to x of our expression for ψ, we find

$$\psi_x = 2xe^{3x}y + 3x^2e^{3x}y + y^3e^{3x} + C'(x).$$

Comparing this with M, we see $C'(x) = 0$, so $C(x) = C_0$ is constant. Taking $C_0 = 0$, then we have the implicit solution

$$x^2ye^{3x} + \frac{y^3}{3}e^{3x} = c.$$

Problem 19. Find the general solution of the following differential equation.

$$y'' + 2y' - 3y = 0$$

Solution. The characteristic equation is $0 = r^2 + 2r - 3 = (r+3)(r-1)$, which has roots $r = -3, 1$. Then the general solution is

$$y = c_1e^{-3t} + c_2e^t.$$

Problem 20.

(a) Solve the following initial value problem.

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, \qquad y'(0) = 0$$

(b) Describe the solution's behavior as $t \to \infty$.

Solution. (a) The characteristic equation is

$$0 = 6r^2 - 5r + 1 = 6r^2 - 2r - 3r + 1 = 2r(3r-1) - (3r-1) = (2r-1)(3r-1)$$

which has roots $r = 1/2, 1/3$. Then the general solution is

$$y = c_1e^{t/2} + c_2e^{t/3}.$$

The initial conditions imply

$$4 = y(0) = c_1 + c_2$$
$$0 = y'(0) = \frac{c_1}{2} + \frac{c_2}{3}.$$

Solving this system, we find $c_1 = -8, c_2 = 12$, which yields the solution to the IVP

$$y = -8e^{t/2} + 12e^{t/3}\,.$$

(b) Since the exponent on $e^{t/2}$ is larger, then $y \to -\infty$ as $t \to \infty$.

Problem 21.

(a) Find the solution to the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2 \qquad y'(0) = 1/2\,.$$

(b) Determine the maximum value of the solution.

Solution. (a) The characteristic equation is

$$0 = 2r^2 - 3r + 1 = 2r^2 - 2r - r + 1 = 2r(r-1) - (r-1) = (2r-1)(r-1)$$

which has roots $1/2, 1$. Then the general solution is

$$y = c_1 e^{t/2} + c_2 e^t\,.$$

The initial conditions imply

$$\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ 1/2 &= y'(0) = c_1/2 + c_2\,. \end{aligned}$$

Solving this system, we find $c_1 = 3$ and $c_2 = -1$, so the solution to the IVP is

$$y = 3e^{t/2} - e^t\,.$$

(b) Setting $y' = \frac{3}{2}e^{t/2} - e^t = 0$, then

$$\frac{3}{2}e^{t/2} = e^t \implies \frac{3}{2} = e^{t/2} \implies \ln(3/2) = t/2 \implies t = 2\ln(3/2)\,.$$

Since $y'' = \frac{3}{4}e^{t/2} - e^t$ and

$$y''(2\ln(3/2)) = \frac{3}{4}\frac{3}{2} - \frac{9}{4} = \frac{9}{8} - \frac{9}{4} = -\frac{9}{8} < 0$$

this is a local maximum. Since this is the only critical value, then it is the global maximum.

Problem 22. Find the Wronskian of the given pairs of functions.

(a) $e^t \sin(t), \quad e^t \cos(t)$
(b) $\cos^2(\theta), \quad 1 + \cos(2\theta)$

Solution. (a)

$$\begin{aligned} W &= \det \begin{pmatrix} e^t \sin(t) & e^t \cos(t) \\ e^t(\sin(t) + \cos(t)) & e^t(\cos(t) - \sin(t)) \end{pmatrix} \\ &= e^{2t}(\sin(t)\cos(t) - \sin^2(t)) - e^{2t}(\cos(t)\sin(t) + \cos^2(t)) \\ &= -e^{2t}(\sin^2(t) + \cos^2(t)) = e^{-2t} \end{aligned}$$

(b) By the double-angle formula, we have

$$1 + \cos(2\theta) = 1 + \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta)$$

since $1 - \sin^2(\theta) = \cos^2(\theta)$. Then

$$\begin{aligned} W &= \begin{pmatrix} \cos^2(\theta) & 2\cos^2(\theta) \\ 2\cos(\theta)\sin(\theta) & 4\cos(\theta)\sin(\theta) \end{pmatrix} \\ &= 4\cos^3(\theta)\sin(\theta) - 4\cos^3(\theta)\sin(\theta) = 0 \end{aligned}$$

Problem 23.

(a) Verify that $y_1(t) = 1$ and $y_2(t) = \sqrt{t}$ are solutions to the differential equation

$$yy'' + (y')^2 = 0$$

for $t > 0$.

(b) Show that $y = c_1 + c_2\sqrt{t}$ is not, in general, a solution to the above equation.

(c) Explain why this does not contradict Theorem 3.2.2.

Solution. (a) The solution $y_1(t) = 1$ trivially satisfies the equation since it yields $0 = 0$. We verify the second solution:

$$y_2y_2'' + (y_2')^2 = t^{1/2} \cdot -\frac{1}{4}t^{-3/2} + \left(\frac{1}{2}t^{-1/2}\right)^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0\,.$$

(b) Let $y = c_1 + c_2t^{1/2}$. Then

$$\begin{aligned} y' &= \frac{c_2}{2}t^{-1/2} \\ y'' &= -\frac{c_2}{4}t^{-3/2} \end{aligned}$$

so

$$\begin{aligned} yy'' + (y')^2 &= (c_1 + c_2t^{1/2}) \cdot -\frac{c_2}{4}t^{-3/2} + \left(\frac{c_2}{2}t^{-1/2}\right)^2 \\ &= -\frac{c_1c_2}{4}t^{-3/2} - \frac{c_2^2}{4}t^{-1} + \frac{c_2^2}{4}t^{-1} = -\frac{c_1c_2}{4}t^{-3/2} \end{aligned}$$

which is 0 if and only if $c_1 = 0$ or $c_2 = 0$.

(c) Since the differential equation is nonlinear, Theorem 3.2.2 does not apply.

Problem 24.

(a) Consider the differential equation

$$y'' + 4y = 0\,.$$

Verify that the functions $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$ are solutions. Do they form a fundamental set of solutions?

(b) Determine a fundamental set of solutions to the following differential equation.

$$y'' + 4y' + 3y = 0$$

Solution. (a) The characteristic equation is $0 = r^2 + 4$ which has roots $r = \pm 2i$. Then we have solutions

$$y_1(t) = e^{2it} = \cos(2t) + i\sin(2t)$$
$$y_2(t) = e^{-2it} = \cos(2t) - i\sin(2t)\,.$$

Taking real and imaginary parts, we have solutions $u(t) = \cos(2t)$ and $v(t) = \sin(2t)$.

The Wronskian

$$W = \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = 2$$

is not identically 0 (in fact, it is never 0), so we have found a fundamental set of solutions.

(b) The characteristic equation is $0 = r^2 + 4r + 3 = (r+3)(r+1)$ which has roots $r = -3, -1$. Thus we have solutions $y_1 = e^{-3t}, y_2 = e^{-t}$. Since the Wronskian

$$W = \det\begin{pmatrix} e^{-3t} & e^{-t} \\ -3e^{-3t} & -e^{-t} \end{pmatrix} = -e^{-4t} + 3e^{-4t} = 2e^{-4t}$$

is never 0, these form a fundamental set of solutions.

Problem 25. For each of the differential equations below, find the general solution.

(a) $y'' + 2y' + 2y = 0$
(b) $y'' + 2y' - 8y = 0$

Solution. (a) The characteristic equation is $0 = r^2 + 2r + 2$ which has solutions

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i\,.$$

Thus the general solution is

$$y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)\,.$$

(b) The characteristic equation is $0 = r^2 + 2r - 8 = (r+4)(r-2)$ which has roots $r = -4, 2$. Thus the general solution is

$$y = c_1 e^{-4t} + c_2 e^{2t}\,.$$

Problem 26. For each of the initial value problems below, find a solution and describe its behavior as $t \to \infty$.

(a) $y'' + 4y = 0, \qquad y(0) = 0, \qquad y'(0) = 1$
(b) $y'' + y' + (5/4)y = 0, \qquad y(0) = 3, \qquad y'(0) = 1$

Solution. (a) The characteristic equation is $0 = r^2 + 4$ which has roots $\pm 2i$. Thus the general solution is

$$y = c_1\cos(2t) + c_2\sin(2t)\,.$$

We compute

$$y' = -2c_1\sin(2t) + 2c_2\cos(2t)\,.$$

The initial conditions imply

$$\begin{aligned} 0 &= y(0) = c_1 \\ 1 &= y'(0) = 2c_2 \end{aligned}$$

so $c_1 = 0$, $c_2 = 1/2$. Thus the solution to the IVP is $y = \dfrac{1}{2}\sin(2t)$. This solution is periodic with period π, and is bounded between $-1/2$ and $1/2$.

(b) The characteristic equation is $0 = r^2 + r + 5/4$ which has roots

$$r = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i\,.$$

Thus the general solution is

$$y = c_1 e^{-t/2}\cos(t) + c_2 e^{-t/2}\sin(t)\,.$$

Note that

$$\begin{aligned} y' &= -c_1\frac{1}{2}e^{-t/2}\cos(t) - c_1 e^{-t/2}\sin(t) - c_2\frac{1}{2}e^{-t/2}\sin(t) + c_2 e^{-t/2}\cos(t) \\ &= e^{-t/2}\left(\left(-\frac{c_1}{2} + c_2\right)\cos(t) + \left(-c_1 - \frac{c_2}{2}\right)\sin(t)\right)\,. \end{aligned}$$

The initial conditions imply

$$\begin{aligned} 3 &= y(0) = c_1 \\ 1 &= y'(0) = -\frac{c_1}{2} + c_2 \end{aligned}$$

so $1 = -3/2 + c_2$, hence $c_2 = 5/2$. Thus the solution to the IVP is

$$y = 3e^{-t/2}\cos(t) + \frac{5}{2}e^{-t/2}\sin(t)\,.$$

Because of the factors of $e^{-t/2}$ in each term, the solution oscillates but with amplitude that decreases exponentially and approaches 0 as $t \to \infty$.