Vibrating Drum

Math 23, Fall 2009

December 1, 2009

Wave equation in $\mathbb{R}^2$

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$u = u(x, y, t)$ transverse displacement
$v =$ wave propagation speed.

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$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \Delta u$$

Laplace's operator

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \left(+ \frac{\partial^2 u}{\partial z^2} \right)$$

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Domain: $D = \{(x, y, t) \mid x^2 + y^2 < 1, t > 0\}$.

Boundary values:

$$u(x, y, t) = 0 \quad \text{for } x^2 + y^2 = 1, t > 0,$$
$$u(x, y, 0) = f(x, y)$$

Polar coordinates

Circular drum $\Rightarrow$ Polar Coordinates

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r \geq 0, \quad 0 \leq \theta < 2\pi$$

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Laplace's operator in polar coordinates

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

BVP in polar coordinates

Wave equation for $u = u(r, t)$ in polar coordinates

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Radially symmetric vibrations $u = u(r, t)$

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Boundary conditions for vibrating drum

$$u(1, t) = 0, \ t \geq 0$$
$$u(r, 0) = f(r), \ 0 \leq r \leq 1$$

Separation of variables

Assume

$$u(r,t) = R(r)T(t)$$

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Obtain two ODEs

$$T'' + \lambda^2 v^2 T = 0 \;\Rightarrow\; T = c_1 \cos(\lambda v t) + c_2 \sin(\lambda v t)$$

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$$r^2 R'' + rR' + \lambda^2 r^2 R = 0, \; R(1) = 0$$

Eigenvalue problem

ODE for R

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Change of variables $s = \lambda r$, $R(r) = R(s/\lambda) = \tilde{R}(s)$.

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Bessel's equation of order zero ($\nu = 0$); solution $\tilde{R}(s) = J_0(s)$

$$R(r) = J_0(\lambda r)$$

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$$R(r) = J_0(\lambda r)$$

Eigenvalue λ is obtained from boundary condition

$$R(1) = J_0(\lambda) = 0$$

Radially symmetric modes

Solution of BVP

$$R(r) = J_0(\lambda r)$$
$$T(t) = \cos(\lambda v t)$$

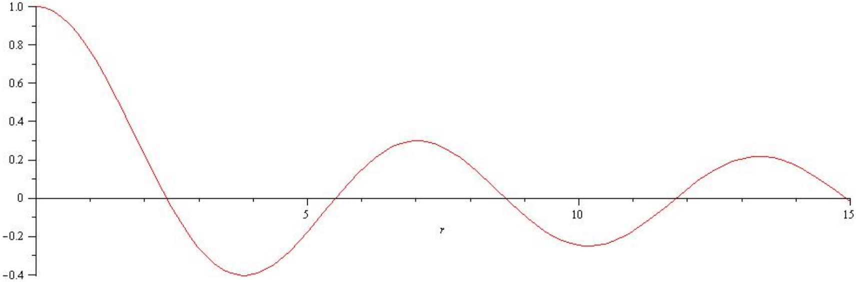
Mode $0k$

$$u(r,t) = J_0(\lambda_{0k} r)\cos(\lambda_{0k} v t)$$

where λ_{0k} is the k-th zero of $J_0(\lambda) = 0$.

Eigenvalues $J_0(\lambda) = 0$

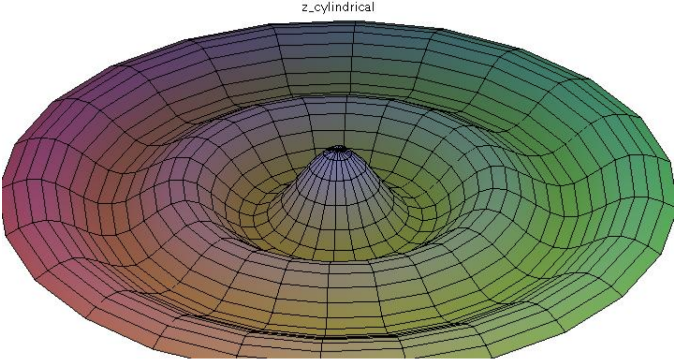
$\lambda_{01} = 2.4048...$, $\lambda_{02} = 5.5201...$, $\lambda_{03} = 8.6537...$,
$\lambda_{04} = 11.7915...$, $\lambda_{05} = 14.9309...$, etc.



Mode 05

$$u(r,\theta,t) = J_0(\lambda_{05} r)\cos(\lambda_{05} vt)$$

Angular frequency $= \lambda_{05} v$, proportional to $\lambda_{05} = 14.9309...$



General case

What if not radially symmetric?

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Three ODEs, two eigenvalue problems (for R and Θ)

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Solutions with eigenvalues n, λ

$$\begin{aligned} \Theta &= \cos(n\theta), \quad n = 0, 1, 2, 3, \cdots \\ R &= J_n(\lambda r), \quad \text{Bessel function of order } n \\ T &= \cos(\lambda v t) \\ u(r, \theta, t) &= J_n(\lambda r)\cos(n\theta)\cos(\lambda v t) \end{aligned}$$

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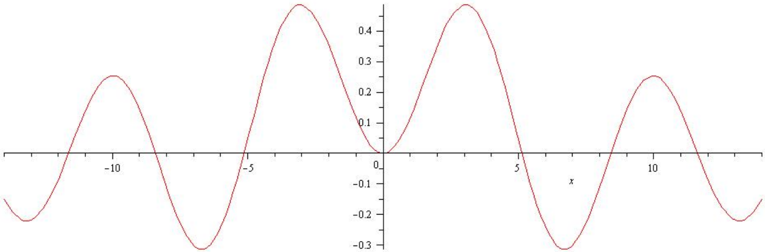
Eigenvalue $\lambda = \lambda_{nk}$ is the k-th zero of $J_n(\lambda) = 0$
Frequency in mode nk is proportional to λ_{nk}.

Modes nk with $n = 2$. At time $t = 1$,

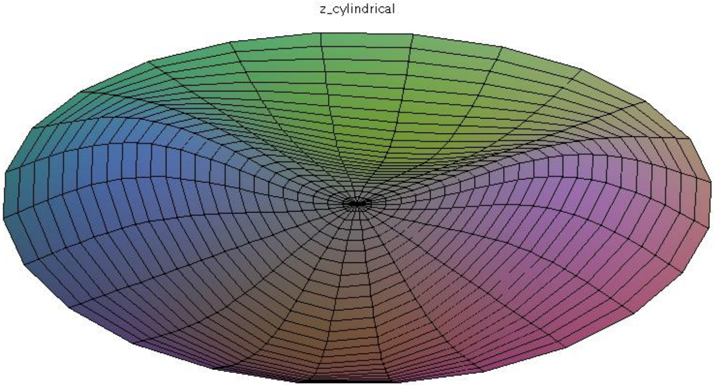
$$u(r,\theta) = J_2(\lambda_{2k} r)\cos(2\theta)$$

$$\lambda_{21} = 5.1356..., \lambda_{22} = 8.4172..., \text{ etc.}$$



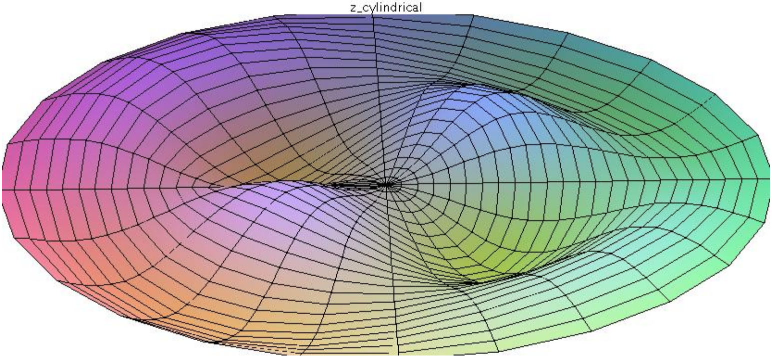
Mode 21

$$u(r,\theta) = J_2(\lambda_{21} r)\cos(2\theta)$$



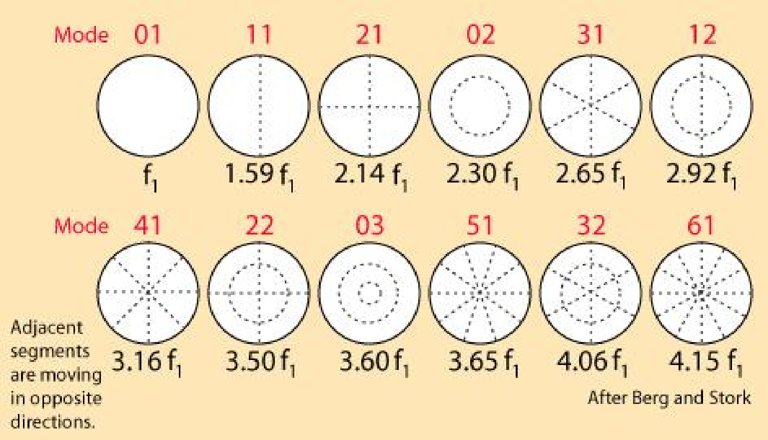
Mode 22

$$u(r,\theta) = J_2(\lambda_{22} r)\cos(2\theta)$$



$$u(r,\theta) = 0 \text{ if } J_2(\lambda_{22} r) = 0 \text{ or } \cos(2\theta) = 0$$

Modes of a vibrating drum



Superposition

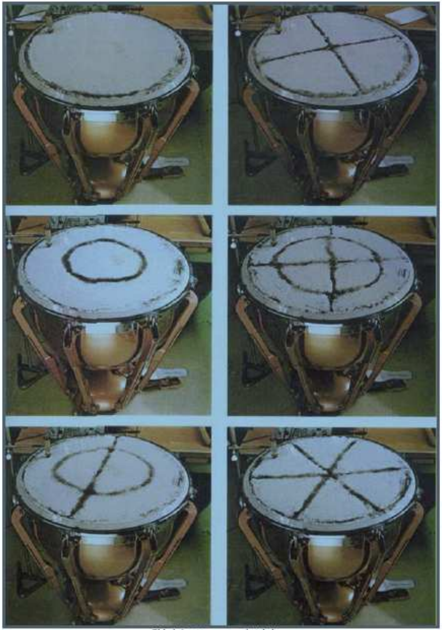
For arbitrary initial value

$$u(r,\theta) = f(r,\theta)$$

vibration of drum = Superposition of modes

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} c_{kn} J_n(\lambda_{nk} r)\cos(n\theta)\cos(\lambda_{nk} v t) + \cdots$$

Coefficients c_{kn} are coefficients in "Fourier-Bessel" series of f.



Chladni patterns on a kettledrum