MATH23 : "Recurring Themes" — ideas for assimilating & grouping material. 11/21/05 Barnett

● Determinant

Do not confuse the 3 times it came up!

i) $\det A = 0 \iff A$ singular, the linear equations $A\vec{x} = \vec{b}$ are either nonunique or unsolvable

ii) Wronskian for 2nd order ODE $y'' + py' + qy = g(t)$, $\left.\begin{matrix} y_1 \\ y_2 \end{matrix}\right\}$ two solutions

$$W[y_1, y_2] = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \underbrace{y_1 y_2' - y_2 y_1'}_{= \text{func of } t.}$$

Note these are equivalent (for n=2 case)

iii) Wronskian for 1st order system of ODEs. $\vec{x}' = A\vec{x}$, $\left.\begin{matrix} \vec{x}^{(1)} \\ \vec{x}^{(2)} \end{matrix}\right\}$ two solutions.

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \det X(t) = \det\begin{bmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{bmatrix} = X_{11}X_{22} - X_{21}X_{12}.$$

for n=2 case.

● Roots of quadratic

i) $\boxed{\text{Ch. 3}}$ $ay'' + by' + cy = 0 \xrightarrow{y = e^{rt}} ar^2 + br + c = 0$ (const-coeff lin 2nd order homogeneous ODE)

3 cases :
- r_1 r_2 real distinct → decay/growth.
- $r = \lambda \pm i\mu$ conjugate complex pair → $e^{\lambda t}(a\cos\mu t + b\sin\mu t)$
- $r_1 = r_2$ repeated root → $(at+b)e^{rt}$

This pattern keeps coming up!

ii) $\boxed{\text{§5-5}}$ Euler eqns. $x^2y'' + \alpha x y' + \beta y = 0 \xrightarrow{y = x^r} r(r-1) + \alpha r + \beta = 0$

Same 3 cases come up. (related to Ch. 3 by $t = \ln x$).

iii) $\boxed{\text{Ch. 7}}$ Eigenvalues of 2-by-2 A matrix in linear(ized) system of ODEs. $\vec{x}' = A\vec{x}$

λ's are roots of quadratic, same 3 cases come up:
- real distinct → sources, sinks, saddles
- complex conj. pair → use $\mathrm{Re}[\vec{\xi}^{(1)} e^{(\lambda + i\mu)t}]$ & $\mathrm{Im}[\text{same}]$ (spirals)
- repeated → improper node, solve for $\vec{\eta}$.

● Existence & Uniqueness.

i) $A\vec{x} = \vec{b}$, solution can
- exist & be unique — A invertible.
- exist & be nonunique } — A singular.
- not exist

Same occurs for boundary-value probs:

ii) $\left.\begin{matrix} u'' + \lambda u = g \\ \text{BCs} \quad u(0) = a, \ u(L) = b \end{matrix}\right\}$
- solution $u(x)$ exists & unique — $\lambda \neq$ eigenvalue $\frac{n^2\pi^2}{L^2}$.
- exists, nonunique } — $\lambda =$ eigenvalue $\frac{n^2\pi^2}{L^2}$.
- not exists