MATH 23 WORKSHEET : Spiral points

Barnett
11/3/05

Find general solution of $\vec{x}' = A\vec{x}$ for $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

$\det \begin{pmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} = \qquad\qquad = 0$ so eigenvalues:

$\lambda_1 =$

$\lambda_2 =$

eigenvectors: at $\lambda = \lambda_1$: $\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\vec{\xi}^{(1)} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

at $\lambda = \lambda_2$: $\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\vec{\xi}^{(2)} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

Gen soln: $\vec{x}(t) =$

How does λ_2 relate to λ_1? $\vec{\xi}^{(2)}$ to $\vec{\xi}^{(1)}$?

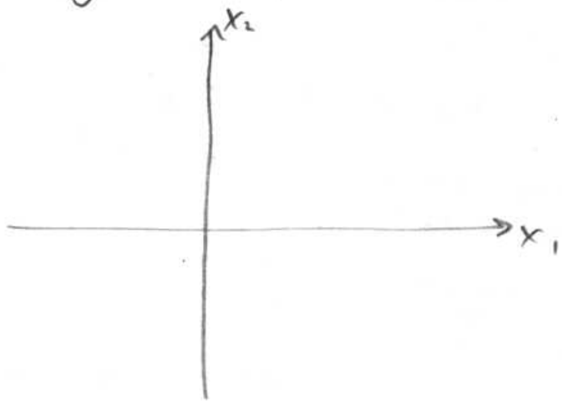
Find $\mathrm{Re}[\, \vec{\xi}^{(1)} e^{r_1 t}] =$

note typo in original : shoulda be 1's not 2's.

$\mathrm{Im}[\, \vec{\xi}^{(1)} e^{r_1 t}] =$

These will be our linearly indep. solutions (if time, check Wronskian ?)

Sketch the trajectories of these solutions :



CW or CCW?

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~ SOLUTIONS ~

Find general solution of $\vec{x}' = A\vec{x}$ for $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

$$\det \begin{pmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 1 + 4 = 0 \qquad \lambda = \tfrac{1}{2}(-2 + \sqrt{4-20})$$

so eigenvalues: $\lambda \pm i\mu$

$\lambda_1 = -1 + 2i$

$\lambda_2 = -1 - 2i$

eigenvectors: at $\lambda = \lambda_1$: $\underbrace{\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}}_{A - \lambda I} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{b}}$

at $\lambda = \lambda_2$: $\begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{a}} - i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{b}}$

Gen soln: (naive complex version) $\vec{x}(t) =$

How does λ_2 relate to λ_1? $\lambda_2 = \overline{\lambda_1}$ conj. $\vec{\xi}^{(2)}$ to $\vec{\xi}^{(1)}$? $\vec{\xi}^{(2)} = \overline{\vec{\xi}^{(1)}}$ ← complex conj.

Find $\mathrm{Re}[\vec{\xi}^{(1)} e^{r_1 t}] = e^{\lambda t}[\vec{a}\cos\mu t - \vec{b}\sin\mu t] = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos\mu t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin\mu t\right] = \begin{pmatrix} e^{-t}\cos\mu t \\ -e^{-t}\sin\mu t \end{pmatrix}$

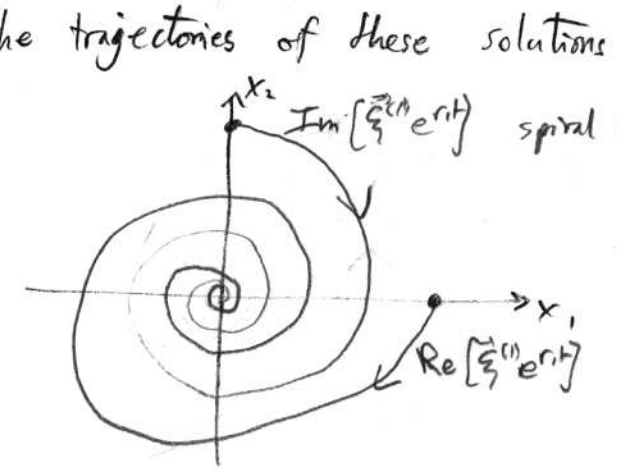
I apologize for typos here: 1 1

$\mathrm{Im}[\vec{\xi}^{(2)} e^{r_2 t}] = e^{\lambda t}[\vec{a}\sin\mu t + \vec{b}\cos\mu t] = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin\mu t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos\mu t\right] = \begin{pmatrix} e^{-t}\sin\mu t \\ e^{-t}\cos\mu t \end{pmatrix}$

it's the Re & Im of the same eigenpair you need.

These will be our linearly indep. solutions (if time, check Wronskian?)

Sketch the trajectories of these solutions:



CW or CCW? clockwise since $\begin{pmatrix} \cos\mu t \\ -\sin\mu t \end{pmatrix}$ goes CW.