MATH 23 WORKSHEET: Separation of Variables. 11/18/05 Barnett

Consider the PDE $t u_{xx} - u_t = 0$

Find separable solutions of the form $u(x,t) = X(x)T(t)$
(use $-\lambda$ as separation constant).

ODE for $X(x)$ is:

write down general solutions.

IF $u(0,t)=0$, $u(L,t)=0$ are BCs, what are allowed λ's?

ODE for $T(t)$ is:

separate T & t to solve it

Combine to write general solution:

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— SOLUTION —

Consider the PDE $t u_{xx} - u_t = 0$

Find separable solutions of the form $u(x,t) = X(x)\,T(t)$
(use $-\lambda$ as separation constant).

$$u_{xx} = X''T$$
$$u_t = XT'$$

PDE: $tX''T = XT'$

ie $\dfrac{X''}{X} = \underbrace{-\lambda}_{\text{sep. const.}} = \dfrac{T'}{tT}$

ODE for $X(x)$ is: $X'' + \lambda X = 0$

write down general solutions. $A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$.

IF $u(0,t) = 0$, $u(L,t) = 0$ are BCs, what are allowed λ's?

As usual, $\sin\frac{n\pi x}{L} \Rightarrow \lambda_n = \frac{n^2\pi^2}{L^2}$ $n=1,2,\dots$

ODE for $T(t)$ is: $T' + \lambda t\, T = 0$ lin. 1st order ODE.

separate T & t to solve it — $p(t) = \lambda t$

int. fac. $\mu(t) = e^{\int p(t)dt} = e^{\frac{\lambda}{2}t^2}$

$$T(t) = \frac{1}{\mu}\left[\int \mu \underbrace{g}_{\text{here } g\equiv 0} dt + c\right] = \frac{c}{\mu(t)} = ce^{-\lambda/2\, t^2}$$

Combine to write general solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{\lambda_n}{2}t^2} \sin\frac{n\pi x}{L}$$

or $\frac{n^2\pi^2}{2L^2}t^2$