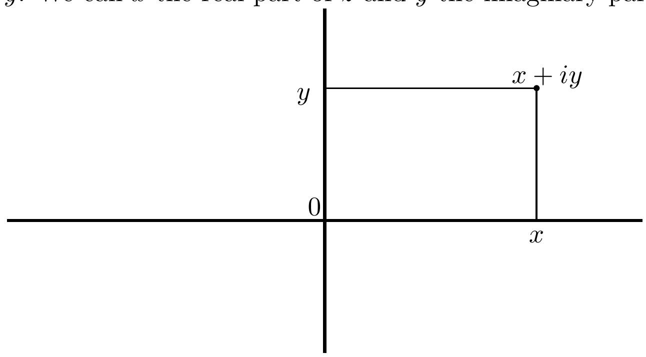
Crash Course in Complex Numbers

Formally a complex number is an ordered pair of real numbers. However we write a complex number z as $z = x + iy$ (or sometimes $z = x + yi$) rather than (x, y). The best way to visualise complex numbers is by thinking of them as points on a plane. The number $z = x + iy$ is the point with horizontal coordinate x and vertical coordinate y. We call x the real part of z and y the imaginary part of z.



Note that any real number a can be thought of a the complex number $a + i0$. We will always omit $0i$ when writing and use i and $-i$ for $1i$ and $-1i$. Thus we can think of i as being the complex number $0+1i$.

Complex numbers add and subtract just as if they were vectors, i.e. $(a+bi)+(c+di) = (a+c)+i(b+d)$. The importance of complex numbers lies in their multiplication which is given by

$$(a + bi) \cdot (c + di) = (ac - bd) + i(bc + ad).$$

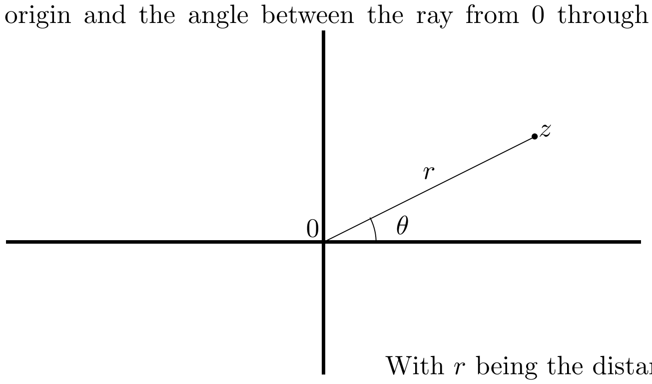
Then it is easy to check that

$$(i)^2 = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + i(0 + 0) = -1.$$

Thus we have a valid square-roots for negative numbers

$$\sqrt{-r} = i\sqrt{r}, \qquad \text{for } r \geq 0.$$

Of course we get a second square-root for -1 by considering $-i$ instead. The multiplication of complex numbers seems much more natural when approached from the perspective of polar coordinates. As any complex number z can be thought of as point in the plane, we can determine z by looking at its distance from the origin and the angle between the ray from 0 through z and the positive x-axis.



With r being the distance from z to 0 we can now write $z = (r\cos\theta) + i(r\sin\theta)$. The distance from z to the origin is called the magnitude (or norm) (or absolute value) of z. It's denoted by $|z|$ and can be computed for $z = x + iy$ as

$$|z| = \sqrt{x^2 + y^2}.$$

Probably the most important formula in the whole subject is the following (known as Euler's formula or sometimes De Moivre's formula)

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

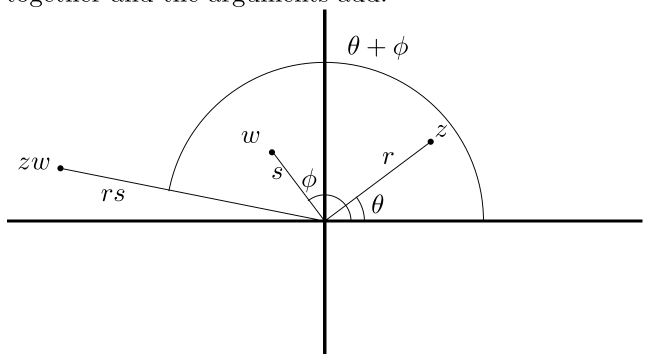
What this means is that in polar coordinates we have the identity

$$z = re^{i\theta}.$$

It is easy to check that $|e^{i\theta}| = 1$ so what this does is split z into magnitude and direction. The angle θ is called the argument of z. Multiplication of complex numbers is now easily interpreted

$$(re^{i\theta}) \cdot (se^{i\phi}) = (rs)e^{i(\theta+\phi)}.$$

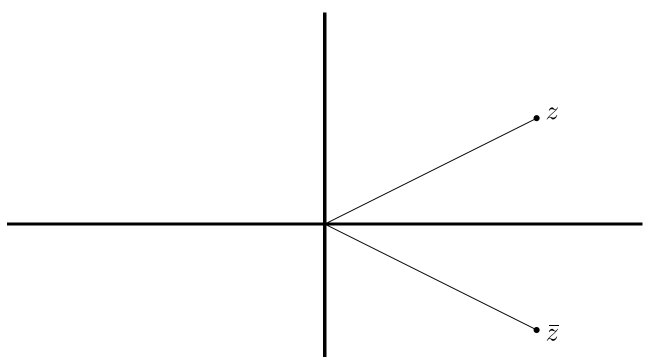
The magnitudes multiply together and the arguments add.



Another useful concept is that of the conjugate of a complex number. For $z = x+iy = re^{i\theta}$ the complex conjugate of z denoted by $\bar{z}$ is given by

$$\bar{z} = x - iy = re^{-i\theta}.$$

Pictorially



Then a simple computation shows that $|z|^2 = z\bar{z}$. Thus dividing by a complex number z is the same as multiplying by $\frac{\bar{z}}{|z|^2}$. For example

$$\frac{3+2i}{1-4i} = (3+2i) \cdot \frac{1+4i}{\sqrt{17}} = \frac{1}{\sqrt{17}}(-5+14i).$$

In addition, conjugates often occur naturally in applications for the following reason:

The roots of a quadratic with real coefficients are either both real or are a pair of conjugate complex numbers.