MATH 22 WORKSHEET. Least squares solution

8/18/06
Bennett
Updated 11/8/16

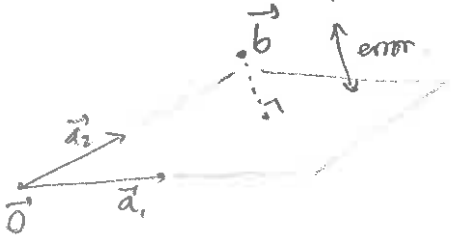
A) Find the least-squares solution to the inconsistent system $A\vec{x} = \vec{b}$ when $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

Form pairs: one compute A^TA, the other A^Tb...

[Hint: your answer involves fractions but no worse than thirds.]

$\hat{x} = \begin{bmatrix} \\ \end{bmatrix}$ Is $\hat{x}$ unique?

B) What is the approximation error? ($\min$ distance from the plane spanned by the columns of A to the point $\vec{b}$)



C) Prove that A^TA is symmetric (is its own transpose)

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SOLUTIONS

A) Find the least-squares solution to the <u>inconsistent</u> system $A\vec{x} = \vec{b}$ when $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Form pairs: one computes A^TA, the other $A^T\vec{b}$...

$$A^TA = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 5 \end{bmatrix}, \quad A^T\vec{b} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

↖ note: always symmetric.

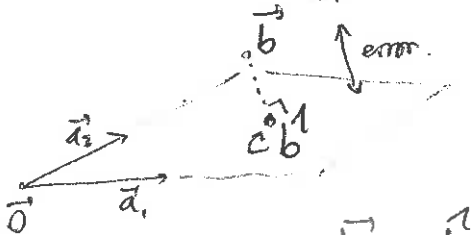
Solve $\left[\begin{array}{cc|c} 8 & 4 & 8 \\ 4 & 5 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 4/3 \end{array}\right]$

or via 2×2 inverse formula.

[Hint: your answer involves fractions but no worse than thirds.]

$\hat{x} = \begin{bmatrix} 1/3 \\ 4/3 \end{bmatrix}$ Is $\hat{x}$ unique? yes; A^TA full rank (because cols of A L.I.; see book)

B) What is the approximation error? ($\min$ distance from the plane spanned by the columns of A to the point $\vec{b}$)



$$\hat{b} = A\hat{x} = \begin{bmatrix} 2/3 \\ 4/3 \\ 10/3 \end{bmatrix}$$

$$\vec{b} - \hat{b} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix} \qquad \|b - \hat{b}\| = \sqrt{2/3} \text{ by Pythag.}$$

C) Prove that A^TA is symmetric (is its own transpose)

$$(A^TA)^T = A^TA^{TT} = A^TA \quad \text{so is own transpose.}$$