Barnett 7/26/17

M22: Review of abstract vector spaces

(without list of problems or tasks you need to know how to solve; make this list!)

A set of vectors, V, is a vector space if it obeys the 10 axioms.

The main ones to check are:
1. $\vec{u},\vec{v}$ in $V \Rightarrow \vec{u}+\vec{v}$ in V
4. there's a $\vec{0}$ vector in V
6. $\vec{u}$ in V, c real $\Rightarrow c\vec{u}$ in V

A subset H of V is a subspace if the above 3 axioms hold with V replaced by H. Since the other axioms are inherited from V, then H is a V.S. in its own right.

examples of subspaces: Span of any vectors in a V.S.; eg Col A in $\mathbb{R}^m$
Nul A, a subspace of $\mathbb{R}^n$
$\mathbb{P}_k$ is a subspace of $\mathbb{P}_q$ for $k \le q$.

Linear independence in a V.S. is just as in $\mathbb{R}^m$, except $c_1\vec{v}_1 + \cdots c_p\vec{v}_p = \vec{0}$ involves the "$\vec{0}$" from axiom 4 above. Eg, $\{1, t, t^2, t^3, \ldots\}$ is L.I.

A basis for a V.S. H is set of vectors $\{\vec{v}_1, \cdots \vec{v}_n\}$ that i) are L.I.
ii) $H = \text{Span}\{\vec{v}_1, \cdots \vec{v}_n\}$

Warning signs: if any vec not in H, their Span may be too big (hence not equal to H).

Spanning set thm: (Thm. 5) "can remove any vec that is L.C. of others w/o affecting a Span". Since can repeat this until L.I., some subset of the vecs is a basis for the span of all of them.

Coordinates of $\vec{x}$ in a basis $\mathcal{B} = \{b_1 \cdots b_n\}$ are $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ in $c_1 b_1 + \cdots c_n b_n = \vec{x}$.

(Thm 8) $\vec{x} \longrightarrow [\vec{x}]_{\mathcal{B}}$ is a linear transformation (↳ by Thm. 7) from $V \rightarrow \mathbb{R}^n$ that is one-to-one & onto.
This "coord. map" is thus an isomorphism, meaning V & $\mathbb{R}^n$ are isomorphic.
This is super useful for proving various theorems; you can treat $a + bt + ct^2$ like $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Dimension of a V.S. V is simply the number of vectors in any basis for V.
If H is subspace of V then $\dim H \le \dim V$. (Thm. 11).

More on coord map:

Thm. 7 says $\vec{x} \to [\vec{x}]_B$ is indeed a map, since there's "unique representation" exactly one $[\vec{x}]_B$ for each $\vec{x}$.

We proved this in lecture! It needs i) & ii) basis properties.

Let's prove parts of Thm 8:

Coord map is onto: let $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ be any point in $\mathbb{R}^n$,

then set $\vec{x} = c_1 b_1 + \cdots c_n b_n$. By construction $\vec{c}$ is the image of $\vec{x}$ under the map. $\square$

Coord map is one-to-one: let $\vec{x}$ and $\vec{y}$ in V both map to $\vec{c}$ in $\mathbb{R}^n$

Then $\vec{x} = c_1 b_1 + \cdots c_n b_n$

& $\vec{y} = c_1 b_1 + \cdots c_n b_n$, so $\vec{x} = \vec{y}$ $\square$

Note both these proofs feel "too easy", because the coord map is a little "backwards", meaning its inverse map $[\vec{x}]_B \to \vec{x}$ is the simpler object, defined by $\vec{x} = c_1 b_1 + \cdots c_n b_n$ where $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

In fact, thm 7 was the harder theorem.

Coord map is linear: you prove part of this in HW5.

You should now make a list of problem types you need to practise solving that goes along with the above theory...