Lecture 29

Math 22 Summer 2017
August 23, 2017

Just for today

- SVD concluded
- Review

SVD spectral-like decomposition

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Let $A = U\Sigma V^T$ be an SVD of A (with rank r).

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$$
\begin{aligned}
A &= U\Sigma V^T \\
&= [\sigma_1 \mathbf{u}_1 \ \cdots \ \sigma_r \mathbf{u}_r \ \mathbf{0} \cdots \mathbf{0}] \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\
&= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T
\end{aligned}
$$

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Each term in this sum is an $m \times n$ matrix of rank 1.

SVD spectral-like decomposition

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Each term in this sum is an $m \times n$ matrix of rank 1. Decomposing A into a sum of rank 1 matrices (ordered by the singular values) is the starting point for applications involving low rank approximations of A.

Low rank approximation example

Recall example ☃:

Low rank approximation example

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$$\underbrace{\begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}}_{A_{3\times 2}} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]} \underbrace{\begin{bmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}}_{\Sigma_{3\times 2}} \underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}^T}_{\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix}}$$

$$= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$$

$$= \begin{bmatrix} 6 & 3 \\ 6 & 3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}$$

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Which rank 1 matrix “best approximates” A?