Lecture 28

Math 22 Summer 2017
August 21, 2017

Just for today

- SVD and applications

§7.4 Proof of Theorem 10

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$$\begin{aligned} AV &= [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_n] \\ &= [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_r \ \underbrace{\mathbf{0} \ \cdots \ \mathbf{0}}_{n-r}] \\ &= [\sigma_1\mathbf{u}_1 \ \cdots \ \sigma_r\mathbf{u}_r \ \underbrace{\mathbf{0} \ \cdots \ \mathbf{0}}_{n-r}] \end{aligned}$$

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$$U\Sigma = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m] \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & \end{array}\right] = [\sigma_1\mathbf{u}_1 \ \cdots \ \sigma_r\mathbf{u}_r \ \underbrace{\mathbf{0} \ \cdots \ \mathbf{0}}_{n-r}]$$

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Thus

$$U\Sigma V^T = (U\Sigma)V^T = (AV)V^T = A(VV^T) = AI_n = A.$$

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How do we find V? The columns of V are normalized eigenvectors of $A^T A$. A basis for $\mathrm{Nul}(A^T A - 90 I_2)$ is $\{[2\ 1]^T\}$.

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Let $A = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}$ Then $A^TA = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}$ Looks symmetric. The charpoly of A^TA is $(\lambda - 90)(\lambda - 10)$, so what are the singular values? $\sigma_1 = \sqrt{90}$, $\sigma_2 = \sqrt{10}$. Thus

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}.$$

How do we find V? The columns of V are normalized eigenvectors of A^TA. A basis for $\mathrm{Nul}(A^TA - 90I_2)$ is $\{[2\ 1]^T\}$. A basis for $\mathrm{Nul}(A^TA - 10I_2)$ is $\{[-1\ 2]^T\}$. Normalizing we get that

$$V = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

§7.4 Example ☃ continued

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$$A\mathbf{v}_1 = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3\sqrt{5} \\ 3\sqrt{5} \\ 0 \end{bmatrix} \implies \mathbf{u}_1 = \frac{A\mathbf{v}_1}{\|A\mathbf{v}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

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For some small 2×2 examples you might want to take a look at `https://goo.gl/oiFXd8`

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Let A have rank r, and consider the SVD:

$$[\mathbf{u}_1 \ \cdots \ \mathbf{u}_r \ \mathbf{u}_{r+1} \ \ldots \ \mathbf{u}_m] \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & \end{array}\right] \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \\ \mathbf{v}_{r+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}$$

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- $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an orthonormal basis of $\mathrm{Col} A$.
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- $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an orthonormal basis of $\mathrm{Col}A$.
- $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ is an orthonormal basis of $(\mathrm{Col}A)^\perp$.
- $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathrm{Nul}A$.

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- $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathrm{Nul}A$.
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal basis of $(\mathrm{Nul}A)^\perp$.

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Suppose $A = U\Sigma V^T$ is an SVD of A. We can use this to find an SVD of A^T. Check that $A^T = V\Sigma^T U^T$ is an SVD of A^T.

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What is the benefit?

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Suppose $A = U\Sigma V^T$ is an SVD of A. We can use this to find an SVD of A^T. Check that $A^T = V\Sigma^T U^T$ is an SVD of A^T.

What is the benefit? If A is $m \times n$, then the SVD of A requires us to orthogonally diagonalize an $n \times n$ matrix $A^T A$.

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What is the benefit? If A is $m \times n$, then the SVD of A requires us to orthogonally diagonalize an $n \times n$ matrix $A^T A$. The SVD of A^T requires us to orthogonally diagonalize an $m \times m$ matrix $(A^T)^T(A^T) = AA^T$.

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Classwork

Find an SVD for

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Let $B = A^T$.
Note $A^T A$ is 3×3 and AA^T is 2×2.
Also note that

$$B^T B = AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.$$

Classwork solutions

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Also, this might be the right time to take a look at `https://goo.gl/oiFXd8`.

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Each term in this sum is an $m \times n$ matrix of rank 1.

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Each term in this sum is an $m \times n$ matrix of rank 1. Decomposing A into a sum of rank 1 matrices (ordered by the singular values) is the starting point for applications involving low rank approximations of A.