

Lecture 28

Math 22 Summer 2017 August 21, 2017



SVD and applications



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Now compute $U\Sigma$:

$$U\Sigma = [\mathbf{u}_1 \cdots \mathbf{u}_m] \begin{bmatrix} \sigma_1 & & \\ & \ddots & 0 \\ & & \sigma_r \\ \hline & 0 & \bullet \end{bmatrix} = [\sigma_1 \mathbf{u}_1 \cdots \sigma_r \mathbf{u}_r \underbrace{\mathbf{0} \cdots \mathbf{0}}_{n-r}]$$



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Thus

$$U\Sigma V^{T} = (U\Sigma)V^{T} = (AV)V^{T} = A(VV^{T}) = AI_{n} = A.$$



§7.4 SVD Example 🗞



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How do we find V? The columns of V are normalized eigenvectors of $A^T A$. A basis for $\operatorname{Nul}(A^T A - 90I_2)$ is $\{[2\ 1]^T\}$. A basis for $\operatorname{Nul}(A^T A - 10I_2)$ is $\{[-1\ 2]^T\}$. Normalizing we get that

$$V = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} 2/\sqrt{5} \ -1/\sqrt{5} \\ 1/\sqrt{5} \ 2/\sqrt{5} \end{bmatrix}.$$





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Is $\{\mathbf{u}_1, \mathbf{u}_2\}$ a basis for \mathbb{R}^3 ? No! We need to extend it to a basis of \mathbb{R}^3 . So what is \mathbf{u}_3 ? $\mathbf{u}_3 = [0 \ 0 \ 1]^T$. Then $U = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]$.

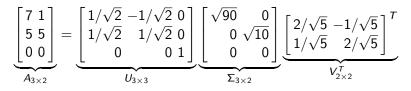




We can now verify that our singular value decomposition of \boldsymbol{A} works:



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We can now verify that our singular value decomposition of *A* works:

$$\underbrace{\begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}}_{A_{3\times 2}} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U_{3\times 3}} \underbrace{\begin{bmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}}_{\Sigma_{3\times 2}} \underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}^{T}}_{V_{2\times 2}^{T}}$$

For some small 2 \times 2 examples you might want to take a look at https://goo.gl/oiFXd8

§7.4 SVD and fundamental spaces



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Let A have rank r,



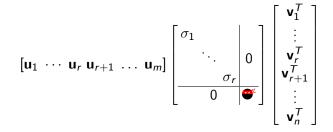
Recall $(\operatorname{Row} A)^{\perp} = \operatorname{Nul} A$ and $(\operatorname{Col} A)^{\perp} = \operatorname{Nul} (A^{\mathsf{T}}).$

Let A have rank r, and consider the SVD:



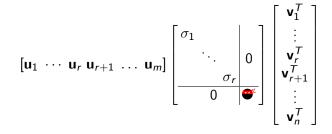
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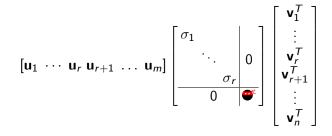


• $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an orthonormal basis of ColA.

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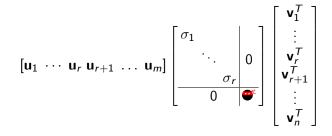


{u₁,...,u_r} is an orthonormal basis of ColA.
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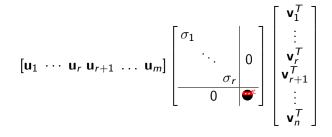


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Classwork



Find an SVD for

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$



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Let
$$B = A^T$$
.
Note $A^T A$ is 3×3 and AA^T is 2×2 .
Also note that

$$B^{\mathsf{T}}B = AA^{\mathsf{T}} = \begin{bmatrix} 17 & 8\\ 8 & 17 \end{bmatrix}.$$





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$$V = \begin{bmatrix} \mathbf{v}_1 \ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

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So now we have V and Σ (for $B = A^T$ remember)

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$$B\mathbf{v}_1 = \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 0 \end{bmatrix}, \quad B\mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ -4/\sqrt{2} \end{bmatrix}$$

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$$B\mathbf{v}_1 = \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 0 \end{bmatrix}, \quad B\mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ -4/\sqrt{2} \end{bmatrix}$$

and $\mathbf{u}_1 = (B\mathbf{v}_1)/\sigma_1, \mathbf{u}_2 = (B\mathbf{v}_2)/\sigma_2.$

 $B^T B$ has eigenvalues $\lambda_1 = 25, \lambda_2 = 9$. So *B* has singular values $\tau_1 = 5, \sigma_2 = 3$. Now find eigenvectors of $B^T B$ corresponding to the 2 eigenvalues. We get $\mathbf{x}_1 = [1 \ 1]^T$ and $\mathbf{x}_2 = [-1 \ 1]^T$. Now we obtain

$$V = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

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and $\mathbf{u}_1 = (B\mathbf{v}_1)/\sigma_1, \mathbf{u}_2 = (B\mathbf{v}_2)/\sigma_2$. How do we get \mathbf{u}_3 ?

Classwork solutions continued





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Then $\mathbf{u}_3 = \mathbf{x} / \|\mathbf{x}\| = [-2/3 \ 2/3 \ 1/3]^T$. So what is the conclusion?

Classwork solutions concluded





$$A^{T} = U\Sigma V^{T}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/(3\sqrt{2}) & -2/3 \\ 1/\sqrt{2} & 1/(3\sqrt{2}) & 2/3 \\ 0 & -4/(3\sqrt{2}) & 1/3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$



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Constrained optimization





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SVD allows us to find an explicit vector \mathbf{x} in the domain of T (subject to the constraint) so that $||A\mathbf{x}||$ is as large as possible.

Also, this might be the right time to take a look at https://goo.gl/oiFXd8.







 $A = U\Sigma V^{T}$ = $[\sigma_{1}\mathbf{u}_{1} \cdots \sigma_{r}\mathbf{u}_{r} \mathbf{0}\cdots\mathbf{0}]\begin{bmatrix}\mathbf{v}_{1}^{T}\\\vdots\\\mathbf{v}_{n}^{T}\end{bmatrix}$ = $\sigma_{1}\mathbf{u}_{1}\mathbf{v}_{1}^{T} + \cdots + \sigma_{r}\mathbf{u}_{r}\mathbf{v}_{r}^{T}$



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Each term in this sum is an $m \times n$ matrix of rank 1. Decomposing A into a sum of rank 1 matrices (ordered by the singular values) is the starting point for applications involving low rank approximations of A.