Lecture 27

Math 22 Summer 2017
August 18, 2017

Just for today

- A bit on least-squares
- SVD

§6.5 Theorem 15

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$$\begin{aligned} A\hat{\mathbf{x}} &= QR\hat{\mathbf{x}} \\ &= QRR^{-1}Q^T\mathbf{b} \\ &= QQ^T\mathbf{b} = \hat{\mathbf{b}}. \end{aligned}$$

□

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$$\begin{aligned}\hat{\mathbf{x}} &= R^{-1}Q^T\mathbf{b} \\ &= \begin{bmatrix} -1/6 & -1/42 & 1/14 & 5/42 \\ 7/6 & 8/21 & -1/7 & -17/42 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 5/14 \\ 2/7 \end{bmatrix}.\end{aligned}$$

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Note that this also shows the unilluminating fact that $(A^TA)^{-1}A^T = R^{-1}Q^T$ when defined.

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$$\sum_{i=1}^{4}(y_i - ct_i - d)^2 = (1-2c-d)^2 + (2-5c-d)^2 + (3-7c-d)^2 + (3-8c-3)^2.$$

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If we let $A = \begin{bmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} c \\ d \end{bmatrix},$

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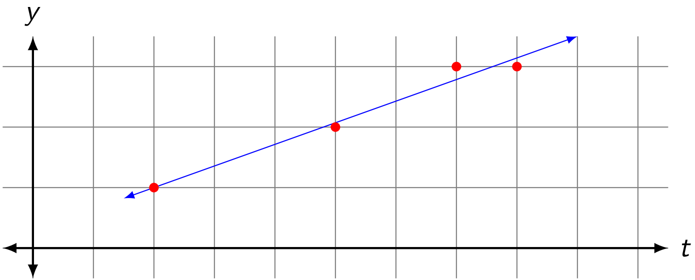

SVD Preliminaries

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Surprisingly the converse is also true! Emma will speak about this (§7.1 Spectral Theorem) in x-hour on Tuesday.

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$$\text{Let } A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}.$$

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Let $A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$. Then

$$A = P \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} P^T$$

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with

$$P = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/2 & 1/2 \\ 1/\sqrt{2} & 0 & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}.$$

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What does this tell us about A?

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Note that for each $\mathbf{v}_i$ we have

$$\begin{aligned} \|A\mathbf{v}_i\|^2 &= (A\mathbf{v}_i) \cdot (A\mathbf{v}_i) \\ &= (A\mathbf{v}_i)^T (A\mathbf{v}_i) \\ &= \mathbf{v}_i^T A^T A \mathbf{v}_i \\ &= \mathbf{v}_i^T B \mathbf{v}_i \\ &= \mathbf{v}_i^T \lambda_i \mathbf{v}_i \\ &= \lambda_i (\mathbf{v}_i \cdot \mathbf{v}_i) = \lambda_i \end{aligned}$$

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Moreover, we can write

$$\underbrace{\sigma_1, \ldots, \sigma_r}_{>0}, \underbrace{\sigma_{r+1}, \ldots, \sigma_n}_{\text{all zero}}.$$

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The index r above turns out to be the rank of the matrix A.

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Thus

$$U\Sigma V^T = (U\Sigma)V^T = (AV)V^T = A(VV^T) = AI_n = A.$$

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How do we find V? The columns of V are normalized eigenvectors of $A^T A$. A basis for $\mathrm{Nul}(A^T A - 90 I_2)$ is $\{[2\ 1]^T\}$. A basis for $\mathrm{Nul}(A^T A - 10 I_2)$ is $\{[-1\ 2]^T\}$.

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How do we find V? The columns of V are normalized eigenvectors of $A^T A$. A basis for $\mathrm{Nul}(A^T A - 90 I_2)$ is $\{[2\ 1]^T\}$. A basis for $\mathrm{Nul}(A^T A - 10 I_2)$ is $\{[-1\ 2]^T\}$. Normalizing we get that

$$V = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

§7.4 Example ☃ continued

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Is $\{\mathbf{u}_1, \mathbf{u}_2\}$ a basis for $\mathbb{R}^3$?

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§7.4 Example ☃ continued

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§7.4 Example ☃ concluded

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We can now verify that our singular value decomposition of A works:

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We will conclude this course by explaining some of the applications of SVD. So stay tuned.

§7.4 Example ☃ concluded

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For some practice with small 2×2 examples you might want to take a look at `https://goo.gl/oiFXd8`