Lecture 23

Math 22 Summer 2017
August 09, 2017

Just for today

- §6.2 Orthogonal sets

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This tells us that $c_1 = 0$. Why? Similarly, we can show all other coefficients are zero. □

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What's the proof?

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What's the proof? Use the boxed equation to rewrite $\mathbf{y} \cdot \mathbf{u}_j$.

§6.2 Classwork

Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}.$$

1. Let $W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Is $\mathbf{u}_3 \in W^\perp$?
2. Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ an orthogonal set?
3. Let $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the coefficients of $\mathbf{y}$ in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. That is, find $c_1, c_2, c_3 \in \mathbb{R}$ so that

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

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Let $W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

1. Find $\hat{\mathbf{y}}$, the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}_2$.
2. What is the distance from $\mathbf{y}$ to the line spanned by $\mathbf{u}_2$?
3. We can also project onto subspaces with dimension greater than 1. Looking ahead to §6.3, the projection of $\mathbf{y}$ onto W is the sum of two projections. Can you see which ones?

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Examples?

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What's the proof?

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What's the proof? What if the columns of U are just orthogonal instead of orthonormal?

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$$U = \begin{bmatrix} 1 & -1/2 & 2/3 \\ 0 & 1 & 2/3 \\ 1 & 1/2 & -2/3 \end{bmatrix}.$$

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$$\begin{bmatrix} 1 & 0 & 1 \\ -1/2 & 1 & 1/2 \\ 2/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 2/3 \\ 0 & 1 & 2/3 \\ 1 & 1/2 & -2/3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}.$$

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Proof.

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T (U\mathbf{y}) = (\mathbf{x}^T U^T)(U\mathbf{y}) = \mathbf{x}^T \underbrace{U^T U}_{I_n} \mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$$

□

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Let

$$U = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{3}} \\ 0 & \frac{2}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{3}} \\ \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{\frac{3}{2}} & -\sqrt{\frac{1}{3}} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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But it is certainly tedious.

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Looking back at U from our example on the previous slide, what do you notice about the rows of U?