Lecture 22

Math 22 Summer 2017
August 07, 2017

Just for today

- §6.1 Inner products and orthogonality
- §6.2 Orthogonal sets

§6.1 Definitions

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The **distance** between $\mathbf{u}$ and $\mathbf{v}$ is defined to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

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$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The inner product on $\mathbb{R}^n$ generalizes these notions of angles and distance to higher dimensions.

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Why does this make sense as a generalization of perpendicular?

§6.1 Pythagorean Theorem

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Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

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Proof.

Expand $\|\mathbf{u}+\mathbf{v}\|^2$ to get

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot(\mathbf{u}+\mathbf{v})+\mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \underbrace{\mathbf{u}\cdot\mathbf{u}}_{\|\mathbf{u}\|^2}+\underbrace{\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}}_{2(\mathbf{u}\cdot\mathbf{v})}+\underbrace{\mathbf{v}\cdot\mathbf{v}}_{\|\mathbf{v}\|^2}$$

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Let W be a subspace of $\mathbb{R}^n$. Let $\mathbf{z} \in \mathbb{R}^n$. We say $\mathbf{z}$ is **orthogonal to** W if $\mathbf{z} \cdot \mathbf{w} = \mathbf{0}$ for every $\mathbf{w} \in W$. We define the **orthogonal complement of** W to be

$$W^{\perp} := \{\mathbf{z} \in \mathbb{R}^n \text{ such that } \mathbf{z} \text{ orthogonal to } W\}.$$

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One direction is obvious. Conversely, suppose $\mathbf{x} \perp \mathbf{w}_i$ and the span of $\{\mathbf{w}_i\}_{i=1}^k$ is W.

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$$\mathbf{x} \cdot \mathbf{w} = \underbrace{\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}}_{1\times n} \underbrace{\begin{bmatrix} \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ \\ \end{bmatrix}}_{n\times k} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}}_{k\times 1}.$$

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Why is this equal to 0? □

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- $W^\perp$ closed under addition:
  Let $\mathbf{z}_1, \mathbf{z}_2 \in W^\perp$. Then for every $\mathbf{w} \in W$ we have
  $(\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{w} = \mathbf{z}_1 \cdot \mathbf{w} + \mathbf{z}_2 \cdot \mathbf{w} = 0 + 0 = 0$.

$\square$

§6.1 Orthogonality and Nul, Row, Col

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Proof.

For the first part show $\subseteq$ and $\supseteq$. For the second part, replace A with A^T to get

$$\Big(\underbrace{\mathrm{Row}\,(A^T)}_{\mathrm{Col}\, A}\Big)^{\perp} = \mathrm{Nul}\,(A^T).$$

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This tells us that $c_1 = 0$. Why? Similarly, we can show all other coefficients are zero. □

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Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *be an orthogonal basis for a subspace* $W \subseteq \mathbb{R}^n$*.*

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Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *be an orthogonal basis for a subspace* $W \subseteq \mathbb{R}^n$*. Then for every* $\mathbf{y} \in W$*, we can write*

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An **orthogonal basis** of a subspaace $W \subseteq \mathbb{R}^n$ is a basis of W that is an orthogonal set.

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with c_j given explicitly by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}.$$

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What's the proof? Use the boxed equation to rewrite $\mathbf{y} \cdot \mathbf{u}_j$.