Lecture 21

Math 22 Summer 2017
August 04, 2017

Just for today

- Applications: Markov chains, PageRank

§4.9 Markov Chains

Definition

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$$S = \begin{bmatrix} .7 & .2 & 0 \\ .1 & .6 & .2 \\ .2 & .2 & .8 \end{bmatrix}, \quad \mathbf{P}_0 = \begin{bmatrix} .1 \\ .5 \\ .4 \end{bmatrix}$$

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How does this help us find the long-term behaviour?

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We should recognize this vector as an element of the $\lambda = 1$ eigenspace scaled to be a probability vector.

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A sufficient condition for an $n \times n$ matrix S to have this property is if we have a strictly decreasing inequality of eigenvalue magnitudes:

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How does this relate to our previous example?

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http://www.ams.org/samplings/feature-column/fcarc-pagerank

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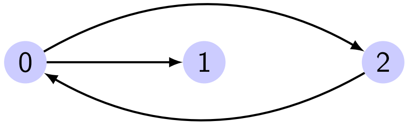

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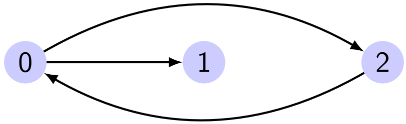

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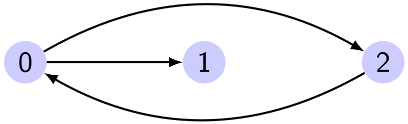

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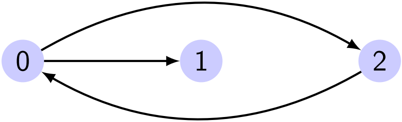

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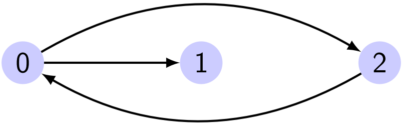

How do we get a stochastic matrix from this? Well, we start with the *weighted adjacency matrix* of the network. This might not be stochastic because what happens when we get to node 2? By picking a new node at random we arrive at a stochastic matrix.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1/3 & 1 \\ 1/2 & 1/3 & 0 \\ 1/2 & 1/3 & 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 4/10 \\ 3/10 \\ 3/10 \end{bmatrix}$$

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$$G = \alpha S + (1 - \alpha)(1/n)\mathbf{1}_n.$$

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Let's finish up our example

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$$G = \alpha \begin{bmatrix} 0 & 1/3 & 1 \\ 1/2 & 1/3 & 0 \\ 1/2 & 1/3 & 0 \end{bmatrix} + (1-\alpha)(1/3) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/20 & 1/3 & 9/10 \\ 19/40 & 1/3 & 1/20 \\ 19/40 & 1/3 & 1/20 \end{bmatrix}$$

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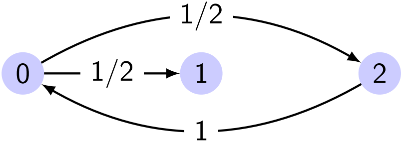
We find that G has steady state vector

$$\mathbf{P} = \begin{bmatrix} 37/94 \\ 57/188 \\ 57/188 \end{bmatrix} = \begin{bmatrix} 0.393617021276596 \\ 0.303191489361702 \\ 0.303191489361702 \end{bmatrix}.$$

PageRank Example concluded

$$\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$



$$S = \begin{bmatrix} 0 & 1/3 & 1 \\ 1/2 & 1/3 & 0 \\ 1/2 & 1/3 & 0 \end{bmatrix} \longrightarrow G = \begin{bmatrix} 1/20 & 1/3 & 9/10 \\ 19/40 & 1/3 & 1/20 \\ 19/40 & 1/3 & 1/20 \end{bmatrix}$$

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$$G = \alpha(H + A) + (1 - \alpha)(1/n)\mathbf{1} = \alpha H + \alpha A + (1 - \alpha)(1/n)\mathbf{1}.$$

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$$G = \alpha\left(\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}\right) + \frac{1-\alpha}{n}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$