Lecture 20

Math 22 Summer 2017
August 02, 2017

Just for today

- §5.3 Diagonalization

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Diagonalizability is closely tied to eigenthings...

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How do you determine diagonalizability when the eigenvalues are not distinct?

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Let's see how we identify and diagonalize matrices in practice...

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3. In this case, D is the diagonal matrix with the eigenvalues down the diagonal and P is the matrix whose columns are the basis vectors of the corresponding eigenspaces. Permuting columns is fine, just make sure P and D correspond to each other.

§5.3 Classwork

1. Explicitly diagonalize the following matrices (if possible).
   (a) $A = \begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$
   (b) $B = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$
   (c) $C = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$
   (d) $D = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$
2. Write an expression for A^k using its diagonal representation.
3. Use the expression for A^k to evaluate $A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.