Lecture 17

Math 22 Summer 2017
July 26, 2017

Just for today

- §4.5 Finish up
- §4.6 Rank

§4.5 Theorem 11

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Note that this proof works in the infinite-dimensional case as well, but requires Zorn's Lemma.

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Corollary of previous theorem and the spanning set theorem. □

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Find the dimension of the subspace H of $\mathbb{R}^3$ defined by

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{r} x + y = 0 \\ y + z = 0 \\ x - z = 0 \end{array} \right\}.$$

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Now, how does this tell us the dimension of H?

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Notice that $\mathrm{Row}A = \mathrm{Col}A^T$.

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Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & -2 & 2 & 3 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Find a basis for $\mathrm{Col} A$. What is the dimension of $\mathrm{Col} A$? What vector space is $\mathrm{Col} A$ a subspace of?
2. Find a basis for $\mathsf{Nul}\, A$. What is the dimension of $\mathsf{Nul}\, A$? What vector space is $\mathsf{Nul}\, A$ a subspace of?
3. Find a basis for $\mathrm{Row} A$. What is the dimension of $\mathrm{Row} A$? What vector space is $\mathrm{Row} A$ a subspace of?

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3. Find a basis for $\mathrm{Row} A$. What is the dimension of $\mathrm{Row} A$? What vector space is $\mathrm{Row} A$ a subspace of?

Can you see a relationship between the dimensions of these spaces that will hold for general A?

§4.6 The rank theorem

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How can we use this theorem?

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(n) $\mathrm{Col} A = \mathbb{R}^n$.

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(p) $\operatorname{rank} A = n$.

(q) $\operatorname{Nul} A = \{\mathbf{0}\}$.

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