# Lecture 15

Math 22 Summer 2017
July 21, 2017

# Just for today

§4.4 Coordinates

- What are coordinates for an element of a vector space?
- How can coordinates be represented geometrically?
- Coordinate maps $V \to \mathbb{R}^n$ and isomorphism

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$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

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### Proof.

Consider another representation and use independence. □

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The map $V \to \mathbb{R}^n$ defined by $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is called a **coordinate map**.

# §4.4 Classwork

Let's see how this works in a concrete example:

`https://math.dartmouth.edu/~m22x17/section2lectures/classwork15.pdf`

Front page only for now!

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*Let* $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *be a basis of a vector space* $V$*. Then the map to coordinates* $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ *is linear, one-to-one, and onto.*

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What's the proof?

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How does the previous theorem apply to $\mathbb{P}_3$?

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Pick a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of $V$
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We define **the matrix of $T$ relative to the bases $\mathcal{B}$ and $\mathcal{C}$, denoted ${}_{\mathcal{C}}[T]_{\mathcal{B}}$** by

$${}_{\mathcal{C}}[T]_{\mathcal{B}} = \Big[\, [T(\mathbf{b}_1)]_{\mathcal{C}} \ [T(\mathbf{b}_2)]_{\mathcal{C}} \ \cdots \ [T(\mathbf{b}_n)]_{\mathcal{C}} \,\Big].$$

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How does this relate to coordinate vectors?

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The matrix ${}_{\mathcal{C}}[\text{id}]_{\mathcal{B}}$ is called the **change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**. Let's see how this works in our classwork example (back page)!

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What is the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$? Well,

$$ {}_{\mathcal{C}}[T]_{\mathcal{B}} = \left[\, [T(1)]_{\mathcal{C}} \;\, [T(t)]_{\mathcal{C}} \;\, [T(t^2)]_{\mathcal{C}} \;\, [T(t^3)]_{\mathcal{C}} \,\right] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}. $$

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and $T(2+3t+4t^2+5t^3) = 3+8t+15t^2$.