Lecture 11

Math 22 Summer 2017
July 12, 2017

Just for today

- “Quiz” today
- §3.1,§3.2 on determinants

§3.1 A_{ij} submatrices

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For example, for $A = \begin{bmatrix} 2 & -1 & 0 & -5 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ we have,

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For example, for $A = \begin{bmatrix} 2 & -1 & 0 & -5 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ we have, $A_{23} = \begin{bmatrix} 2 & -1 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$.

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The determinant of an $n \times n$ matrix $A = (a_{ij})$ is given by:

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}.\end{aligned}$$

§3.1 Cofactor expansion

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This equation is an example of **cofactor expansion along the first row of** A and is just a restatement of our definition of $\det A$.

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We see that $\det A = 4$.

§3.1 Theorem 1

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$$\begin{aligned} \det A &= (-1)^{1+2} a_{12} \det A_{12} + (-1)^{2+2} a_{22} \det A_{22} + (-1)^{3+2} a_{32} \det A_{32} \\ &= 4. \end{aligned}$$

Let $A = \begin{bmatrix} 4 & -1 & 5 & 1 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$

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§3.1 Theorem 2

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Theorem

If A is a triangular matrix, the $\det A$ *is the product of the entries on the main diagonal of A.*

§3.2 Theorem 3

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$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{bmatrix}, D = \begin{bmatrix} 2 & 4 & 8 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{bmatrix}, E = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 0 & -4 \\ 0 & -1 & 1 \end{bmatrix}$.

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We find that $\det B = \det C = \det A$, $\det D = 2 \det A$, and $\det E = -\det D = -2 \det A$.

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A square matrix A is invertible if and only if $\det A \neq 0$.

§3.2 Theorem 5

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Theorem

If A is an $n \times n$ matrix, the $\det A^T = \det A$.

§3.2 Theorem 6

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Theorem

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.