Lecture 10X

Math 22 Summer 2017
July 11, 2017

Reminders/Announcements

- "Quiz" Wednesday to practice for Midterm
- Midterm will cover material through §2.3 (Today)

Just for today

- §2.3 Characteristics of invertible matrices

§2.3 The invertible matrix theorem

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(a) A is an invertible matrix.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) The matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(g) $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There is an $n \times n$ matrix C such that $CA = I_n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

(l) A^T is an invertible matrix.

§2.3 Invertibe linear transformations

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Definition

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A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$
$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

§2.3 Theorem 9

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Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is an invertible matrix. Moreover, if T is invertible, then the map $S : \mathbb{R}^n \to \mathbb{R}^n$ defined by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique inverse linear transformation of T.

§2.3 Proof of Theorem 9

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($\Rightarrow$): Suppose T is invertible. By the IMT, it suffices to show T is onto. To do this, let $\mathbf{b} \in \mathbb{R}^n$ (codomain of T). Then $S(\mathbf{b}) \in \mathbb{R}^n$ (domain of T) maps to $\mathbf{b}$ under the map T.

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(uniqueness): Suppose S' is another inverse of T. Let $\mathbf{b} \in \mathbb{R}^n$ be arbitrary.

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($\Leftarrow$): Suppose A is invertible and let $S(\mathbf{x}) = A^{-1}\mathbf{x}$. then verify that S works as an inverse of T.

($\Rightarrow$): Suppose T is invertible. By the IMT, it suffices to show T is onto. To do this, let $\mathbf{b} \in \mathbb{R}^n$ (codomain of T). Then $S(\mathbf{b}) \in \mathbb{R}^n$ (domain of T) maps to $\mathbf{b}$ under the map T.

(uniqueness): Suppose S' is another inverse of T. Let $\mathbf{b} \in \mathbb{R}^n$ be arbitrary. By the IMT there exists $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$ (T is onto).

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(uniqueness): Suppose S' is another inverse of T. Let $\mathbf{b} \in \mathbb{R}^n$ be arbitrary. By the IMT there exists $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$ (T is onto). But this shows that $S(\mathbf{b}) = S'(\mathbf{b})$ since

$$S(\underbrace{T(\mathbf{x})}_{\mathbf{b}}) = S'(\underbrace{T(\mathbf{x})}_{\mathbf{b}}) = \mathbf{x}.$$

Since $\mathbf{b}$ was arbitrary, we see that $S(\mathbf{b}) = S'(\mathbf{b})$ for all $\mathbf{b} \in \mathbb{R}^n$.

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($\Rightarrow$): Suppose T is invertible. By the IMT, it suffices to show T is onto. To do this, let $\mathbf{b} \in \mathbb{R}^n$ (codomain of T). Then $S(\mathbf{b}) \in \mathbb{R}^n$ (domain of T) maps to $\mathbf{b}$ under the map T.

(uniqueness): Suppose S' is another inverse of T. Let $\mathbf{b} \in \mathbb{R}^n$ be arbitrary. By the IMT there exists $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$ (T is onto). But this shows that $S(\mathbf{b}) = S'(\mathbf{b})$ since

$$S(\underbrace{T(\mathbf{x})}_{\mathbf{b}}) = S'(\underbrace{T(\mathbf{x})}_{\mathbf{b}}) = \mathbf{x}.$$

Since $\mathbf{b}$ was arbitrary, we see that $S(\mathbf{b}) = S'(\mathbf{b})$ for all $\mathbf{b} \in \mathbb{R}^n$. That is, they are the same. □

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§2.3 Proof of IMT

- $(a) \implies (j) \implies (d) \implies (c) \implies (b) \implies (a)$
- $(a) \implies (k) \implies (g) \implies (a)$
- $(g) \iff (h) \iff (i)$
- $(d) \iff (e) \iff (f)$
- $(a) \iff (l)$