Lecture 09

Math 22 Summer 2017
July 10, 2017

Reminders/Announcements

- X-hour Tuesday (§2.3)
- "Quiz" Wednesday to practice for Midterm
- Midterm will cover material through §2.3

Just for today

- Practice problems about inverses
- More about matrices: §2.1/§2.2 powers and transpose
- §1.10 Linear difference equations

§2.2 Classwork

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- Find the sequence of elementary matrices that transform the above A to I_2.

§2.1 Powers of a matrix

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What is the k-th power of a *diagonal matrix*?

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What's an example?

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You can check the first 3 as an exercise, but let's look at the last part together.

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Writing individual matrix entries more concisely we have:

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How can we generalize this to finish the proof?

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- $$AB = \begin{bmatrix} \text{row}_1(A)\cdot B \\ \vdots \\ \text{row}_m(A)\cdot B \end{bmatrix}$$
- $$AB = \begin{bmatrix} A\cdot \text{col}_1(B) & \cdots & A\cdot \text{col}_p(B) \end{bmatrix}.$$

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$$A^T (A^{-1})^T = (A^{-1}A)^T = (I_n)^T = I_n.$$

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§1.10 Linear difference equations

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Then we call the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ together with the *transition matrix* A a **linear difference equation**.

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What is the transition matrix A that describes this system as a linear difference equation?

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`http://sagecell.sagemath.org/?q=hgoihs`