

Lecture 08

Math 22 Summer 2017 July 07, 2017



- Answers to classwork07 from last time: How can you check if a map is linear?
- Recall matrix multiplication and associativity
- §2.2 elementary matrices and inverses

Answers to classwork07





https://math.dartmouth.edu/~m22x17/section2lectures/ classwork07ans.pdf

Review matrix multiplication and associativity





Recall matrix multiplication via dot products.



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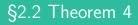


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§2.2 Theorem 4









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§2.2 Theorem 5





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Proof.

Show that $A^{-1}\mathbf{b}$ is a solution. Show that any solution must be equal to $A^{-1}\mathbf{b}$.

§2.2 Theorem 6







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What is the inverse of A⁻¹?
(b) Suppose A and B are invertible. Then AB is invertible and (AB)⁻¹ = B⁻¹A⁻¹.



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Proof.

Check the definition.

§2.2 Elementary matrices





Definition

An **elementary matrix** is a matrix that is obtained by performing a single row operation to the identity matrix I_n .

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Performing a row operation to an $m \times n$ matrix A is equivalent to left multiplication by an $m \times m$ elementary matrix E created by performing the desired row operation to I_m .



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Every elementary matrix E is invertible, and E^{-1} is the elementary matrix that transforms E back to the identity.



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§2.2 Theorem 7





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§2.2 Algorithm to compute A^{-1}





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Theorem

Let A be an $n \times n$ matrix. Consider the augmented matrix $[A I_n]$. If A is row equivalent to I_n , then $[A I_n]$ is row equivalent to $[I_n A^{-1}]$. If the RREF of A is not I_n , then A is not invertible.

As an example,

$$\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & -1 & -4 & 1 \\ 0 & 0 & 1 & 2 & 6 & -1 \end{bmatrix}$$

§2.2 Classwork





Find
$$A^{-1}$$
 for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.





Find the sequence of elementary matrices that transform the above A to I_2 .