Lecture 08

Math 22 Summer 2017

July 07, 2017

Just for today

- Answers to classwork07 from last time: How can you check if a map is linear?
- Recall matrix multiplication and associativity
- §2.2 elementary matrices and inverses

Answers to classwork07

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https://math.dartmouth.edu/~m22x17/section2lectures/classwork07ans.pdf

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Today we will define the inverse of a matrix. To do this we need an *identity* for the operation. What is the identity for $+$? What is the identity for $\cdot$? What is the identity for matrix multiplication? What are examples of inverses in the familiar context of $+$ and $\cdot$?

§2.2 The inverse of a matrix

Definition

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The real number $ad - bc$ is an example of a **determinant** in the 2×2 case. Example?

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Proof.

Show that $A^{-1}\mathbf{b}$ is a solution.
Show that any solution must be equal to $A^{-1}\mathbf{b}$. $\square$

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Proof.
Check the definition. □

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An **elementary matrix** is a matrix that is obtained by performing a single row operation to the identity matrix I_n.

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Every elementary matrix E is invertible, and E^{-1} is the elementary matrix that transforms E back to the identity.

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If A is invertible, then why does A have a pivot in every row? A square matrix with a pivot in every row is row equivalent to the identity matrix.

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Let A be an $n \times n$ matrix. Consider the augmented matrix $[A \; I_n]$. If A is row equivalent to I_n, then $[A \; I_n]$ is row equivalent to $[I_n \; A^{-1}]$. If the RREF of A is not I_n, then A is not invertible.

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As an example,

$$\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & -1 & -4 & 1 \\ 0 & 0 & 1 & 2 & 6 & -1 \end{bmatrix}.$$

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- Use A^{-1} from the previous part to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- Find the sequence of elementary matrices that transform the above A to I_2.