Lecture 07

Math 22 Summer 2017
July 05, 2017

Just for today

- Answers to classwork06 from last time
- §1.9 more about linear maps

Answers to classwork06

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https://math.dartmouth.edu/~m22x17/section2lectures/classwork06ans.pdf

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OK, enough smiley faces for one day.

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Can you make any observations about how the matrix A relates to where the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are mapped by T?

We summarize this in the following theorem.

§1.9 Theorem 10

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The matrix A is called the ***standard matrix for*** *T.*
Sometimes we write $[T]$ to indicate the standard matrix for T.

§1.9 Proof of Theorem 10

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What about "uniqueness"? □

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Solution:

$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

§1.9 injectivity and surjectivity

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If T is both injective and surjective we say that T is a **bijection**.

How can we algorithmically determine if linear maps have these properties?

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Let T have standard matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -7 & 11 & -13 \\ 0 & 0 & 0 & 17 \end{bmatrix}.$$

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What is the domain/codomain of T? $T : \mathbb{R}^4 \to \mathbb{R}^3$.

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What is the domain/codomain of T? $T : \mathbb{R}^4 \to \mathbb{R}^3$.
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What is the domain/codomain of T? $T : \mathbb{R}^4 \to \mathbb{R}^3$.
Is T onto? Yes! No matter what $\mathbf{b} \in \mathbb{R}^3$ we choose, the augmented matrix $[A|\mathbf{b}]$ corresponds to a consistent system.

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Can you see an easy way to define a linear transformation T that is a *bijection*?

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Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map.* T *is one-to-one if and only if* $T(\mathbf{x}) = \mathbf{0}$ *has only the trivial solution.*

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Proof.

First note that $T(\mathbf{0}) = \mathbf{0}$.

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Can you see why we are done? □

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Proof.

We saw that the columns are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Then:

(a) *T is onto if and only if the columns of A span $\mathbb{R}^m$.*

Proof.

The image of T is the span of the columns. □

(b) *T is one-to-one if and only if the columns of A are linearly independent.*

Proof.

We saw that the columns are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Now apply previous theorem. □

Classwork

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https://math.dartmouth.edu/~m22x17/section2lectures/classwork07.pdf