Lecture 06

Math 22 Summer 2017
July 03, 2017

Just for today

- Review slash finish up §1.7 on linear independence
- §1.8 Linear transformations

§1.7 Linear Independence

Definition

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Not necessarily! Example?

§1.7 Classwork

Find all values of $h \in \mathbb{R}$ for which the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$$

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So the set is dependent if and only if $h = 6$.

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More precisely, for $\mathbf{x} \in \mathbb{R}^n$ and $A_{m\times n}$ matrix, we define a map $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) := A\mathbf{x}$.

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Linear maps preserve the algebraic operations of addition and scalar multiplication.

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$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

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How would this question change if A were not in REF?

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https://math.dartmouth.edu/~m22x17/section2lectures/classwork06.pdf